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حول بعض الفئات الجزئية من الدوال الميرومورفية المرتبطة بمؤثر نور التكاملي

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On Certain Subclasses of Meromorphic Functions Associated with Noor Integral Operator

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الملخص

يتناول هذا البحث دراسة بعض الفئات الجزئية من الدوال الميرومورفية المرتبطة بمؤثر نور التكاملي. يركز العمل على استكشاف خصائص التبعية (Subordination) والالتفاف (Convolution)لهذه الفئات، وذلك من خلال بناء تعريفات دقيقة للفئات محل الدراسة وربطها بمؤثر نور.

يعتمد البحث على مجموعة من المبرهنات والنتائج التكميلية التي تعزز الإطار النظري وتوضح كيفية ارتباط هذه الفئات بالخواص الرياضية الأساسية في مجال الدوال التحليلية والميرومورفية. كما يتضمن العمل مقدمات نظرية (Lemmas) تمثل أساسًا لإثبات النتائج الرئيسية، إضافة إلى عدد من النتائج الخاصة (Corollaries) التي توضح حالات تطبيقية للفئات المدروسة.

تكمن أهمية هذا البحث في كونه يقدم إسهامًا علميًا في مجال نظرية الدوال الميرومورفية عبر ربطها بمؤثرات تكاملية حديثة، مما يفتح آفاقًا جديدة لدراسة البنى الرياضية ذات الصلة وإيجاد تطبيقات أوسع في التحليل المركب.

الكلمات الدالة: الدوال التحليلية، الدوال الميرومورفية، مؤثر نور التكاملي، الالتفاف.

Abstract

This paper investigates certain subclasses of meromorphic functions associated with the Noor integral operator. The study focuses on exploring the properties of subordination and convolution for these subclasses, through precise definitions and their connection to the Noor operator.

The work is supported by a set of theorems and complementary results that strengthen the theoretical framework and clarify how these subclasses are related to fundamental properties in the theory of analytic and meromorphic functions. The paper also includes preliminary lemmas that form the foundation for proving the main results, along with several corollaries that illustrate specific cases and applications of the studied subclasses.

The significance of this research lies in its contribution to the field of meromorphic function theory by linking it with modern integral operators, thereby opening new avenues for the study of related mathematical structures and broader applications in complex analysis.

Keywords: Analytic functions, Meromorphic functions, Noor integral operator, Convolution.

1. Introduction

Putting ∇ denote the class of functions as follows:

$$\zeta(\nu) = \frac{1}{\nu} + \sum_{k=0}^{\infty} a_k \nu^k \tag{1.1}$$

which are holomorphic in the punctured unit disk $U^* = \{\nu: 0 < |\nu| < 1\} = U \backslash \{0\}$, with a simple pole at the origin.

If $\zeta(\nu)$ and $g(\nu)$ are holomorphic in U, we recall that $\zeta(\nu)$ is subordinate to $g(\nu)$, written $\zeta \prec g$ or $\zeta(\nu) \prec g(\nu)$ $(\nu \in U)$, if there exists a Schwarz function $\Psi(\nu)$ in U with $\Psi(0) = 0$ and $|\Psi(\nu)| < 1$ $(\nu \in U)$, such that $\zeta(\nu) = g(\Psi(\nu))$ $(\nu \in U)$. If $g(\nu)$ is univalent in U, then the following equivalence relationship holds true:

$$\zeta(\nu) \prec g(\nu) \quad (\nu \in U) \Leftrightarrow \zeta(0) = g(0) \text{ and } \zeta(U) \subset g(U).$$

If functions $\zeta(\nu) \in \nabla$ given by (1.1) and $g(\nu) \in \nabla$ given by

$$g(\nu) = \frac{1}{\nu} + \sum_{k=0}^{\infty} a_k \nu^k \tag{1.2}$$

the convolution product of $\zeta(\nu)$ and $g(\nu)$ is given by

$$(\zeta * g)(\nu) := \frac{1}{\nu} + \sum_{k=0}^{\infty} a_k \nu^k =: (g * \zeta)(\nu) \tag{1.3}$$

For $\zeta(\nu) \in \nabla$ give by (1,1), denote $D^n\zeta(\nu): \nabla \rightarrow \nabla$ the operator defined by

$$D^n\zeta(\nu) = \frac{1}{\nu(1-\nu)^{n+1}} * \zeta(\nu)$$

The operator $D^n\zeta(\nu)$ was introduced by Uralegaddi and Somanatha [1] and studied by Cho [2]. This operator is called the Ruscheweyh derivative of order n^{th} of ζ .

Analogous to operator $D^n\zeta(\nu)$, we here define an integral operator $I_n: \nabla \to \boldsymbol{\nabla}$ as follows:

Set $\zeta_n(\nu) = \frac{1}{\nu(1-\nu)^{n+1}}$, $(n \in \mathbb{N}_0 , \nu \in U^*)$, and let $\zeta_n(\nu)^{(\dagger)}$ be defined sueh that

$$\zeta_n(\nu) * \zeta_n{}^{(\dagger)}(\nu) = \frac{1}{\nu(1-\nu)^2}$$

We have,

$$I_n\zeta(\nu) = \zeta_n^{(\dagger)}(\nu) * \zeta(\nu)$$

$$\Longrightarrow I_n\zeta(\mathrm{v}) = \frac{1}{\mathrm{v}} + \sum_{k=0}^{\infty} \frac{n!\,(k+2)!}{(n+k+1)!}\; a_k\, \mathrm{v}^k$$

$$\mathrm{v}\big(I_n\,\zeta(\mathrm{v})\big)' = n\, I_{n-1}\zeta(\mathrm{v}) - (n+1)I_n\zeta(\mathrm{v}) \qquad (1.4)$$

Now, we introduce the following subclasses of ∇associated with the Noor integral operator $I_n\zeta(\nu)$

Definition 1. For fixed parameters A , B $(-1 \le B < A \le 1)$, a function $\zeta(\nu) \in \nabla$ is said to be in the class $\nabla_n(\lambda, A, B)$ if

$$-\mathrm{v}^2\left\{\left(1-\frac{1}{2\lambda}\right)\big(I_n\zeta(\mathrm{v})\big)' + \frac{1}{2\lambda}\big(I_{n-1}\zeta(\mathrm{v})\big)'\right\} \prec \frac{1+A\mathrm{v}}{1+B\mathrm{v}} \quad (\mathrm{v}\epsilon\, U^*) \qquad (1.5)$$

Where $n \in \mathbb{N}$ and $\lambda > 0$.

In this paper, we derive some subordination results for the class $\nabla_n(\lambda, A, B)$ and investigate several convolution properties of functions which have been defined here by means of the Noor integral operator $I_n\zeta(\nu)$

2. Preliminaries

To prove our main result, we need the following lemmas.

"**Lemma 1** ([3]; see also [4]). Let $\phi(\nu)$ be analytic in U and $h(\nu)$ be analytic and convex (univalent) in U with $h(0) = \phi(0) = 1$.

If

$$\phi(\nu)+\frac{\nu\phi'(\nu)}{\gamma}\prec h(\nu) \quad (Re(\gamma)\geq 0;\ \gamma\neq 0; \nu\in U), \tag{2.1}$$

then

$$\phi(\nu)\prec\psi(\nu)=\gamma\nu^{-\gamma}\int_0^{\nu}t^{\gamma-1}h(t)dt\prec\ h(\nu) \quad (\nu\in U),$$

and $\psi(\nu)$ is the best dominant of (2.1).

We denote by $P(\gamma)$ the class of functions $\varphi(\nu)$ given by

$$\varphi(\nu)=1+b_1\nu+b_2\nu^2+\ldots\ , \tag{2.2}$$

Which are analytic in U and satisfy the following inequality:

$$Re\left(\varphi(\nu)\right)>\gamma \quad (0\leq\gamma<1, \nu\in U)."$$

"Lemma 2 ([5]). Let the function $\varphi(\nu)$ given by (2.2) be in the class$P(0)$. Then

$$Re\left(\varphi(\nu)\right)\geq\frac{1-|\nu|}{1+|\nu|} \quad (\nu\in U)."$$

"Lemma 3 ([6]). If $\varphi_j\in P(\gamma_j)$ $\left(0\leq\gamma_j<1; j=1,2\right)$, then $\varphi_1*\varphi_2\in P(\gamma_3)$, $\gamma_3=1-2(1-\gamma_1)(1-\gamma_2)$. The result is best possible "

"Lemma 4 ([7]). For real or complex numbers $a\ ,b\ and\ c\ (\ c\neq 0,-1\ ,-2,\ldots)$,

$$\int_0^1 t^{b-1}\,(1-t)^{c-b-1}(1-\nu t)^{-a}dt$$
$$=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2^1F(a,b,c;\nu) \quad (Re(c)>Re(b)>0); \tag{2.3}$$

$${}_2^1F(a,b,c;\nu)=(1-\nu)^{-a}\ {}_2^1F\left(a,c-b;c;\frac{\nu}{\nu-1}\right), \tag{2.4}$$

and

$${}_2^1F\left(a,b;\frac{a+b+1}{2};\frac{1}{2}\right)=\frac{\sqrt{\pi}\,\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}." \tag{2.5}$$

3. The main results

Unless otherwise mentioned, we assume throughout this work that

$$n \in \mathbb{N},\ \lambda > 0 \text{ and } -1 \leq B < A \leq 1.$$

Theorem 1. Let $f(\nu)$ defined by (1.1) be in the class $\nabla_{\mathrm{n}}(\lambda, A, B)$; then

$$-\nu^2\big(I_n\zeta(\nu)\big)' \prec \mathrm{Q}(\nu) \prec \frac{1+A\nu}{1+B\nu} \quad (\nu \in U^*), \tag{3.1}$$

Where the function $\mathrm{Q}(\boldsymbol{\nu})$ given by

$$\mathrm{Q}(\nu) = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1+B\nu)^{-1}\ {}_2^1F\left(1, 1, 2n\lambda+1; \dfrac{B\nu}{B\nu+1}\right) & (B \neq 0) \\ 1 + \dfrac{2n\lambda}{1+2n\lambda} A\nu & (B = 0), \end{cases}$$

is the best dominant of (3.1). Furthermore,

$$Re\,\{\nu^2(I_n\zeta(\nu))'\} > \rho \qquad (\nu \in U^*), \tag{3.2}$$

where

$$\rho(\lambda, A, B) = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1-B)^{-1}\ {}_2^1F\left(1, 1, 2n\lambda+1; \dfrac{B}{B-1}\right) & (B \neq 0) \\ 1 - \dfrac{2n\lambda}{1+2n\lambda} A & (B = 0), \end{cases}$$

The result is the best possible.

Proof. set

$$\phi(\nu) = -\nu^2(I_n\zeta(\nu))' \quad (\nu \in U^*). \tag{3.3}$$

Then the function $\phi(\nu)$ is holomorphic in U with $\phi(0) = 1$. Differentiating (3.3) and with the aid of the identity (1.4), we get

$$-\nu^2\big(I_{n-1}\zeta(\nu)\big)' = \frac{\nu\phi'(\nu) + n\phi(\nu)}{n}$$

Now by (1.5), we have

$$-\nu^2\left\{\left(1 - \frac{1}{2\lambda}\right)\big(I_n\zeta(\nu)\big)' + \frac{1}{2\lambda}\big(I_{n-1}\zeta(\nu)\big)'\right\} = \phi(\nu) + \frac{\nu\phi'(\nu)}{2n\lambda} \prec \frac{1+A\nu}{1+B\nu} \quad (\nu \in U^*). \tag{3.4}$$

Now, by using lemma 1 for $\gamma = 2n\lambda$, we deduce that

$$\phi(\nu) \prec \mathrm{Q}(\nu) = 2n\lambda\, \nu^{-2n\lambda} \int_0^{\nu} t^{2n\lambda-1} \left(\frac{1+At}{1+Bt}\right) dt$$

$$= \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1+B\nu)^{-1}\, {}_2^1F\left(\left(1, 1, 2n\lambda+1; \dfrac{B\nu}{B\nu+1}\right)\right. & (B \neq 0) \\ 1 + \dfrac{2n\lambda}{1+2n\lambda} A\nu & (B = 0), \end{cases}$$

by a change of variables followed by the use of identities (2.3) and (2.4) (with $a = 1$,

$$b = 2n\lambda, c = b + 1).$$

This prove the assertion (3.1) of Theorem 1.

Next, to prove the assertion (3.2) of Theorem 1, it suffices to show that

$$\inf_{|\nu|<1} \{Re(\mathrm{Q}(\nu))\} = \mathrm{Q}(-1). \tag{3.5}$$

we know that $\phi(\nu) \prec \mathrm{Q}(\nu)$, this means that

$$\phi(\nu) = \mathrm{Q}(\Psi(\nu)), \qquad (|\Psi(\nu)| < 1)$$

$$\Rightarrow Re(\phi(\nu)) = Re\left(\mathrm{Q}(\Psi(\nu))\right)$$

$$\geq \inf_{|\nu|<1} \{Re[\mathrm{Q}(\Psi(\nu))]\} = \mathrm{Q}(-1)$$

$$\Rightarrow Re(\phi(\nu)) \geq \mathrm{Q}(-1)$$

we obtained the assertion (3.5). The result in (3.2) is best possible as the function $\mathrm{Q}(\nu)$ is the best dominant (3.1).

Remark 1. The left hand side of (1.5) is equivalent to

$$-\nu^2 \left\{\left(1 + \frac{1}{n\lambda}\right)\left(I_n\zeta(\nu)\right)' + \frac{z}{2n\lambda} \left(I_n\zeta(\nu)\right)''\right\}$$

so (1.5) is equivalent to

$$-\nu^2 \left\{\left(1 + \frac{1}{n\lambda}\right)\left(I_n\zeta(\nu)\right)' + \frac{z}{2n\lambda} \left(I_n\zeta(\nu)\right)''\right\} \prec \frac{1+A\nu}{1+B\nu} \tag{3.6}$$

Now putting $\lambda = \frac{\sigma}{1-2\sigma}$ $(0 < \sigma < \frac{1}{2})$ in Theorem 1, we get the following result:

Corollary 1. If $\zeta(z) \in \boldsymbol{\nabla}$ satisfies

$$-\nu^2\left\{\left(1+\frac{1-2\sigma}{n\sigma}\right)\left(I_n\zeta(\nu)\right)' + \frac{1-2\sigma}{2n\sigma}\, z\left(I_n\zeta(\nu)\right)'' \prec \frac{1+A\nu}{1+B\nu}\right\} \quad (\nu \in U^*), \qquad (3.7)$$

then

$$-\nu^2\left(I_n\zeta(\nu)\right)' \prec \mathrm{Q}(\boldsymbol{\nu}) \prec \frac{1+A\nu}{1+B\nu} \qquad (\nu \in U^*), \qquad (3.8)$$

where the function $\mathrm{Q}(\boldsymbol{\nu})$ given by

$$\mathrm{Q}(\nu) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+B\nu)^{-1}\,{}_2^1F\left(1,1,\frac{2n\sigma}{1-2\sigma}+1;\ \frac{B\nu}{B\nu+1}\right) & (B \neq 0) \\ 1 + \frac{2n\sigma}{1+2(n-1)\sigma}\,A\nu & (B = 0), \end{cases}$$

is the best dominant of (3.1). Furthermore,

$$Re(-\nu^2(I_n\zeta(\nu))') > \rho \quad (\nu \in U^*), \qquad (3.9)$$

where

$$\rho(\lambda, A, B) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1}\,{}_2^1F\left(1,1,\frac{2n\sigma}{1-2\sigma}+1;\ \frac{B}{B-1}\right) & (B \neq 0) \\ 1 - \frac{2n\sigma}{1+2(n-1)\sigma}\,A & (B = 0). \end{cases} \qquad (3.10)$$

The result is best possible.

Taking $A = 1-2\delta\ (0 \le \delta < 1), B = -1$ and $\lambda = \frac{1}{2n}$ in Theorem 1, we have:

Corollary 2. If $\zeta(\nu) \in \boldsymbol{\nabla}$, satisfies the following inequality:

$$Re\left\{-\nu^2\left[3\left(I_n\zeta(\nu)\right)' + z\left(I_n\zeta(\nu)\right)''\right]\right\} > \delta \qquad (0 \le \delta < 1\,; \nu \in U^*),$$

then

$$Re\left\{-\nu^2\left(I_n\zeta(\nu)\right)'\right\} > 1 + 2(1-\delta)(\ln 2 - 1) \qquad (\nu \in U^*). \qquad (3.11)$$

The result is best possible.

Taking $A = 1-2\delta\ (0 \le \delta < 1),\ B = -1$ and $\lambda = \frac{1}{4n}$ inTheorem1,and using (2.5), we have:

Corollary 3. If $\zeta(\nu) \in \boldsymbol{\nabla}$, satisfies the following inequality:

$$Re\left\{-\nu^2\left[5\left(I_n\zeta(\nu)\right)' + 2z\left(I_n\zeta(\nu)\right)''\right]\right\} > -\frac{\pi-2}{4-\pi} \tag{3.12}$$

then

$$Re\left\{-\nu^2\left(I_n\nu(\nu)\right)'\right\} > 0 \tag{3.13}$$

The result is best possible.

Theorem 2. If $\zeta(\nu) \in \boldsymbol{\nabla}$ satisfies

$$\nu\left\{\left(1-\frac{1}{2\lambda}\right)\left(I_n\zeta(\nu)\right) + \frac{1}{2\lambda}\left(I_{n-1}\zeta(\nu)\right)\right\} \prec \frac{1+A\nu}{1+B\nu} \quad (\nu \in U^*), \tag{3.14}$$

then

$$\nu I_n\zeta(\nu) \prec \mathrm{Q}(\nu) \prec \frac{1+A\nu}{1+B\nu} \quad (\nu \in U^*),$$

and

$$Re\left(\nu I_n\zeta(\nu)\right) > \rho \quad (\nu \in U^*),$$

where $\mathrm{Q}(\nu)$ and ρ are given as in Theorem 1. The result is best possible.

Proof. Replace $\phi(\nu)$ by $\nu I_n\zeta(\nu)$ in (3.3) and follow the lines of the Theorem 1.

Theorem 3. Let $-1 \leq B_j < A_j \leq 1$ $(i = 1, 2)$. If each of the functions $\zeta_j(\nu) \in \boldsymbol{\nabla}$ satisfies the following subordination condition:

$$\nu\left\{\left(1-\frac{1}{2\lambda}\right)I_n\zeta_j(\nu) + \frac{1}{2\lambda}I_{n-1}\zeta_j(\nu)\right\} \prec \frac{1+A\nu}{1+B\nu} \quad (j = 1, 2;\ \nu \in U^*). \tag{3.15}$$

and if $\zeta(\nu) \in \boldsymbol{\zeta}$ is defined by

$$I_n\zeta(\nu) = I_n\zeta_1(\nu) * I_n\zeta_2(\nu) \tag{3.16}$$

then

$$Re\left[\left(1-\frac{1}{2\lambda}\right)\nu\, I_n\zeta(\nu) + \frac{\nu}{2\lambda}I_{n-1}\zeta(\nu)\right] > \gamma \tag{3.17}$$

where

$$\gamma = 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left[1 - \frac{1}{2} {}_2^1F\left(1, 1, 2n\lambda + 1; \frac{1}{2}\right)\right].$$

The result is best possible when $B_1 = B_2 = -1$.

Proof. Suppose that each of the function $\zeta_j(\nu) \in \nabla$ $(j = 1,2)$, satisfies the condition (3.15). Then, on letting

$$\varphi_j(\nu) = \left(1 - \frac{1}{2\lambda}\right)\nu\, I_n\zeta_j(\nu) + \frac{\nu}{2\lambda} I_{n-1}\zeta_j(\nu) \quad (j = 1,2) \tag{3.18}$$

we have

$$\varphi_j(\nu) \in P(\gamma_j)\left(\gamma_j = \frac{1 - A_j}{1 - B_j}, j = 1, 2\right).$$

from (3.18), we have

$$I_n\zeta_j(\nu) = 2n\lambda\, \nu^{-2n\lambda} \int_0^{\nu} t^{2n\lambda - 1}\varphi_j(t)dt \quad (j = 1, 2). \tag{3.19}$$

Now if $\zeta(\nu) \in \boldsymbol{\nabla}$ is defined by (3.16), we find from (3.19) that

$$I_n\zeta(\nu) = I_n\zeta_1(\nu) * I_n\zeta_2(\nu)$$

$$= \left(2n\lambda\, \nu^{-2n\lambda} \int_0^{\nu} t^{2n\lambda-1}\varphi_1(t)dt\right) * \left(2n\lambda\, \nu^{-2n\lambda} \int_0^{\nu} t^{2n\lambda-1}\varphi_2(t)dt\right)$$

$$= 2n\lambda\, \nu^{-2n\lambda} \int_0^{\nu} t^{2n\lambda-1}\varphi_0(t)dt, \tag{3.20}$$

where

$$\varphi_0(t) = 2n\lambda\, \nu^{-2n\lambda} \int_0^{\nu} t^{2n\lambda-1}(\varphi_1 * \varphi_2)(t)dt \tag{3.21}$$

since $\varphi_1(z) \in P(\gamma_1)$ and $\varphi_2(z) \in P(\gamma_2)$, it follows from Lemma 3 that

$$(\varphi_1 * \varphi_2)(\nu) \in P(\gamma_3) \quad (\gamma_3 = 1 - 2(1 - \gamma_1)(1 - \gamma_2). \tag{3.22}$$

According to Lemma 2, we have

$$Re(\varphi_1 * \varphi_2)(\nu) \geq \gamma_3 + (1 - \gamma_3)\frac{1 - |\nu|}{1 + |\nu|} = (2\gamma_3 - 1) + \frac{2(1 - \gamma_3)}{1 + |\nu|}. \tag{3.23}$$

Now by using (3.23) in (3.21), we get

$$Re\left\{\left(1-\frac{1}{2\lambda}\right)\nu I_n\zeta(\nu)+\frac{z}{2\lambda}I_{n-1}\zeta(\nu)\right\}=Re\big(\varphi_0(\nu)\big)$$

$$=2n\lambda\int_0^1 u^{2n\lambda-1}\,Re\{(\varphi_1*\varphi_2)(u\nu)\}du$$

$$\geq 2n\lambda\int_0^1 u^{2n\lambda-1}\left(2\gamma_3-1+\frac{2(1-\gamma_3)}{1+u|\nu|}\right)du$$

$$>2n\lambda\int_0^1 u^{2n\lambda-1}\left(2\gamma_3-1+\frac{2(1-\gamma_3)}{1+u}\right)du$$

$$=1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1-2n\lambda\int_0^1 u^{2n\lambda-1}(1+u)^{-1}du\right]$$

$$=1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1-\frac{1}{2}\,{}_2^1F\left(1,1,2n\lambda+1;\frac{1}{2}\right)\right]$$

$$=\gamma\quad(\nu\in U^*)$$

where $2\gamma_3-1=1-4\left(\frac{A_1-B_1}{1-B_1}\right)\left(\frac{A_2-B_2}{1-B_2}\right)$ and $2(1-\gamma_3)=4\left(\frac{A_1-B_1}{1-B_1}\right)\left(\frac{A_2-B_2}{1-B_2}\right)$ which completes the proof of assertion (3.17).

For $B_1=B_2=-1$, we consider the functions $\zeta_j(\nu)\in\boldsymbol{\nabla}$ $(j=1,2)$ defined by

$$I_n\zeta_j(\nu)=2n\lambda\,\nu^{-2n\lambda}\int_0^{\nu}t^{2n\lambda-1}\left(\frac{1+A_j(t)}{1-t}\right)dt\quad(j=1,2).$$

for which we have

$$\varphi_j(\nu)=\left(1-\frac{1}{2\lambda}\right)\nu I_n\zeta_j(\nu)+\frac{\nu}{2\lambda}I_{n-1}\zeta_j(\nu)=\frac{1+A_j(\nu)}{1-\nu}\qquad(j=1,2)$$

and

$$(\varphi_1*\varphi_2)(\nu)=\left(\frac{1+A_1\nu}{1-\nu}\right)*\left(\frac{1+A_2\nu}{1-\nu}\right)$$

$$=1+\frac{(1+A_1)(1+A_2)\nu}{1-\nu}.$$

Hence, for $\zeta(\nu)\in\boldsymbol{\nabla}$ given by (3.16), we obtain

$$\left\{\left(1-\frac{1}{2\lambda}\right)\nu I_n\zeta(\nu)+\frac{\nu}{2\lambda}I_{n-1}\zeta(\nu)\right\}=\varphi_0(\nu)$$

$$=2n\lambda\int_0^1 u^{2n\lambda-1}\left(1-(1+A_1)(1+A_2)+\frac{(1+A_1)(1+A_2)}{1-u\nu}\right)du$$

$$=1-(1+A_1)(1+A_2)+(1+A_1)(1+A_2)(1-\nu)^{-1}\,{}_2^1F\left(1,1,2n\lambda+1;\frac{\nu}{\nu-1}\right)$$

$$\longrightarrow 1-(1+A_1)(1+A_2)+\frac{1}{2}\,(1+A_1)(1+A_2)\,{}_2^1F\left(1,1,2n\lambda+1;\frac{1}{2}\right)$$

as $\nu \longrightarrow -1$

which evidently completes the proof of Theorem 3.

Letting $A_J = 1-\eta_j, B_j = -1\ (j = 1,2)$ and $\lambda = \tau$ in Theorem 3, we get the following result:

Corollary 4. If $\zeta(\nu) \in \boldsymbol{\nabla}$ satisfies

$$\nu\left\{\left(1+\frac{1}{2n\tau}\right)I_n\zeta_j(\nu)+\frac{\nu}{2n\tau}\left(I_n\zeta_j(\nu)\right)'\right\}>\eta_j \quad (j=1,2,\nu\in U^*)$$

then

$$Re\left\{\left(1+\frac{1}{2n\tau}\right)\nu\left(I_n\zeta_1(\nu)*I_n\zeta_2(\nu)\right)+\frac{\nu^2}{2n\tau}\left(I_n\zeta_1(\nu)*I_n\zeta_2(\nu)\right)'\right\} > \gamma, \quad (3.24)$$

where

$$\gamma = 1-4(1-\eta_1)(1-\eta_2)[1-\frac{1}{2}\,{}_2^1F\left(1,1,2n\tau+1;\frac{1}{2}\right).$$

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