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Oscillation Criteria for Fractional-Order Third-Order Neutral Differential Equations with Damping

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معايير التذبذب للمعادلات التفاضلية المحايدة من الرتبة الثالثة ذات الرتبة الكسرية مع التخميد

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Abstract

This paper establishes new oscillation criteria for a class of third-order neutral fractional differential equations with nonlinear damping. By employing a modified fractional Riccati transformation combined with integral averaging techniques and the Caputo fractional derivative, we derive sharp sufficient conditions ensuring oscillation of all solutions. The obtained results generalize and improve several known oscillation criteria for integer-order and fractional-order equations reported in the literature. An illustrative example is presented to demonstrate the applicability of the theoretical findings..

Keywords: Fractional differential equations; Neutral equations; Oscillation; Damping; Riccati transformation.

الملخص:

تتناول هذه الورقة البحثية وضع معايير جديدة للتذبذب لفئة من المعادلات التفاضلية الكسرية المحايدة من الرتبة الثالثة ذات التخميد غير الخطي. ومن خلال توظيف تحويل ريكاتي الكسري المعدَّل بالاقتران مع تقنيات المتوسطات التكاملية ومشتقة كابوتو الكسرية، تم اشتقاق شروط كافية دقيقة تضمن تذبذب جميع

الحلول. وتُعد النتائج المتحصل عليها تعميمًا وتحسينًا لعدد من معايير التذبذب المعروفة للمعادلات ذات الرتبة الصحيحة والرتبة الكسرية الواردة في الأدبيات العلمية. كما قُدِّم مثال توضيحي لبيان قابلية تطبيق النتائج النظرية المتوصل إليها.

الكلمات المفتاحية: لمعادلات التفاضلية الكسرية؛ المعادلات المحايدة؛ التذبذب؛ التخميد؛ تحويل ريكاتي.

Introduction:

Oscillation theory constitutes a central topic in the qualitative analysis of differential equations and plays a crucial role in the modeling of physical, engineering, and biological phenomena. In particular, oscillatory behavior characterizes a wide range of processes arising in mechanics, control theory, population dynamics, and signal processing. In recent years, fractional differential equations have attracted considerable attention due to their ability to capture memory and hereditary effects that cannot be adequately described by classical integer-order models. Consequently, substantial efforts have been devoted to extending qualitative theories, including oscillation theory, from integer-order differential equations to their fractional counterparts. Neutral differential equations form an important subclass of functional differential equations, in which the derivative of the unknown function depends explicitly on delayed arguments. The simultaneous presence of neutral terms and fractional-order derivatives substantially increases analytical complexity and limits the applicability of standard oscillation techniques. Although oscillation criteria for third-order neutral differential equations of integer order have been extensively investigated (see, for example, [5, 6, 7]), corresponding results for fractional-order equations remain relatively scarce. Existing studies are often confined to lower-order equations, non-neutral structures, or linear damping terms, leaving a noticeable gap in the theory. Motivated by these observations, the present paper establishes new oscillation criteria for a class of third-order neutral fractional differential equations with nonlinear damping. The analysis is based on a modified fractional Riccati transformation combined with integral averaging techniques. The obtained conditions not only extend several known results for integer-order equations but also improve existing fractional oscillation criteria by incorporating neutral terms and nonlinear damping effects. Consider the third-order neutral fractional differential equation

2. Problem Formulation and Assumptions

Consider the third-order neutral fractional differential equation

$$^{c}D_t^{\alpha}[x(t)+\rho x(t-\tau)]+\gamma(t)(x'(t)+\rho x'^{(t-\tau)}))^{p}+q(t)x^{p}=0,\ \ t\geq t_0 \quad (1)$$

Where

$\alpha \in (2,3], p>1, \rho \in (0,1)$, and $\tau>0$.

For convenience, defined $\quad y(t)=x(t)+\rho x(t-\tau). \quad (2)$

The coefficient $\gamma(t)$, and $q(t)$ are assumed to be continuous on $[t_0,\infty)$.

And cD_t^{α} denotes the Caputo fractional derivative of order $2<\alpha\leq 3$.

Core Assumptions:

(A1) The functions $\gamma, q \in C([t_0, \infty), (0, \infty))$.

(A2) The delay function $\tau(t) = t - \tau \leq t \; and \; \lim_{t\to\infty} \tau(t) = \infty$.

(A3) The nonlinear terms $(x'(t) + \rho x'^{(t-\tau)}))^p$ and x^p satisfy $p > 1$.

(A4) there exist $T_0 \geq t_0$ such that

$\int_{T_0}^{\infty} q(t)\gamma(t)^{-1/p} dt = \infty$.

(A5) $0 < \rho < 1$.

(A6) We assume that $y \in AC^2[t_0, \infty)$ to ensure the validity of

$$\frac{d}{dt}[\,{}^{c}D_t^{\alpha-1}U(t)] = cD_t^{\alpha}U(t)$$

Assumption (A4) is imposed to exclude eventually monotone positive solutions and is standard in the oscillation theory of higher-order differential equations.

3. Auxiliary Results and Riccati Transformation Methodology

This section establishes the foundational definitions, lemmas, and the core transformation technique necessary for the analysis of oscillatory behavior in Equation (1).

3.1 Basic Definitions

We begin by recalling the essential definitions from fractional calculus that underpin this work.

Definition 3.1 (Caputo Fractional Derivative).

Let $\alpha > 0$ with $n - 1 < \alpha \leq n, n \in N$ and $x: [t_0, \infty) \to \mathbb{R}$ be a function such that $x^{(n)}$ is absolutely continuous. The Caputo fractional derivative of order α of $x(t)$ is defined by

$$ {}^{c}D_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-s)^{n-\alpha-1} x^{(n)}(s)ds, \qquad (3)$$

Where $\Gamma(\cdot)$ denotes the Gamma function.

In this study, we assume that the solutions of Equation (1) belong to the space of functions $y \in AC^2[t_0, \infty)$, and $cD_t^{\alpha-1}y \in AC[t_0, \infty)$, where AC^2 denotes the set of functions whose second-order derivatives are absolutely continuous. This regularity assumption is essential to ensure the validity of the fundamental identity:

$$\frac{d}{dt}[\,{}^{c}D_t^{\alpha-1}U(t)] = cD_t^{\alpha}U(t), \quad \text{for } \alpha \in (2, 3].$$

Definition 3.2 (Oscillatory Solution).

A solution $x(t)$ of Equation (1) is said to be **oscillatory** if it has arbitrarily large zeros; otherwise, it is called **non-oscillatory**.

3.2 Behavior of Non-Oscillatory Solutions

The analysis of oscillation criteria typically proceeds via a contradiction argument, assuming the existence of a non-oscillatory solution. Without loss of generality, we assume the existence of an eventually positive solution $x(t) > 0$ and, consequently by (A2), $x(\tau(t)) > 0$ for $t \geq T_1 \geq t_0$.

From the definition of $y(t)$ in (2) and assumption (A1), it follows that

$y(t) > 0$ for $t \geq T_1$.

The structure of Equation (1) imposes specific sign patterns on the associated function $U(t) = \gamma(t)(y'(t))^p$, which are critical for applying the Riccati technique.

3.3 Lemma

Let x(t) be an eventually positive solution of Equation (1), and define

$$y(t) = x(t) + \rho(t)x\big(\tau(t)\big), \quad \mathrm{U(t)} = \gamma(t)\big(y'(t)\big)^p.$$

Then the function U(t) satisfies exactly one of the following two cases for all sufficiently large t:

i. $\mathrm{U(t)} > 0$ and ${}^{c}D_t^{\alpha-1}\mathrm{U(t)} \geq 0, \ \ for\ all\ t \geq T.$ or
ii. $\mathrm{U(t)} < 0 \ \ for\ all\ t \geq T.$

Proof:

Assume that $\mathbf{x(t)}$ is an eventually positive solution of Equation (1)

Then there exist $T_1 \geq t_0$. Such that

$\mathrm{x(t)} > 0$ and $\mathrm{x(\tau(t)} > 0$ for all $t \geq T_1$.

From equation (1), assumptions (A_1) and (A_3), and the definition of U(t), we obtain

$$cD_t^{\alpha}U(t) = {}^{c}D_t({}^{c}D_t^{\alpha-1}U(t)) \leq -q(t)f\big(y(t)\big) \leq -\beta q(t)y^p(t) < 0$$

Consequently, the function $cD_t^{\alpha-1}U(t)$ is non-increasing on $[T_2, \infty)$, for some $T_2 \geq T_1$. Since $\mathrm{U(t)} = \gamma(t)\big(y'(t)\big)^p$ and $\gamma(t) > 0$ by (A_1), the continuity of $y'(t)$ implies that U(t)

Is continuous on $[T_2, \infty)$. Hence there exists $T \geq T_2$ such that U(t) has a fixed sign on $[T, \infty)$.

If $\mathrm{U(t)} < 0 \ \ for\ all\ t \geq T$, then case (ii) holds.

Otherwise, for all $t \geq T$.

In this case, since $D_t^{\alpha}\mathrm{U(t)} \leq 0$, the function $D_t^{\alpha-1}\mathrm{U(t)}$ is non-increasing on $[T, \infty)$.

Therefor,

${}^{c}D_t^{\alpha-1}\mathrm{U(t)} \geq 0$, for all $t \geq T$.

This complete the proof.

3.3 Modified Fractional Riccati Transformation

In this section, we introduce a modified fractional Riccati-type transformation tailored for the neutral fractional differential equation (1). The goal of this transformation is to reduce the order of the fractional derivative and simplify the expression to facilitate the application of differentiation and subsequent analysis of oscillation criteria.

We recall the established auxiliary function from Lemma 3.3:

$$U(t) = \gamma(t)(y'(t))^p$$

We define the modified fractional Riccati transformation W(t) as:

$$\mathrm{W(t)} = \eta(t)\frac{cD_t^{\alpha-1}U(t)}{y(t)^p} \qquad \textit{(4)}$$

where $\eta(t) > 0$ is an arbitrary, positive, and continuously differentiable weight function to be chosen later, and $y(t) = x(t) + \rho(t)x(\tau(t))$.

Applying the standard derivative $\frac{d}{dt}$ to $W(t)$:

$$\frac{d}{dt}\mathrm{W(t)} = \frac{d}{dt}\left[\eta(t)\frac{cD_t^{\alpha-1}U(t)}{y(t)^p}\right]$$

Using the product and quotient rules:

$$\frac{d}{dt}\mathrm{W(t)} = \frac{\eta'(t)cD_t^{\alpha-1}U(t)}{y(t)^p} + \eta(t)\frac{\frac{d}{dt}\left(cD_t^{\alpha-1}U(t)\right).y(t)^p - cD_t^{\alpha-1}U(t).py(t)^{p-1}y'}{[y(t)^p]^2}.$$

Since

$$\frac{d}{dt}\left(cD_t^{\alpha-1}U(t)\right) = cD_t^{\alpha}U(t)$$

and dividing the numerator of the second term by $y(t)^p$, we get:

$$\frac{d}{dt}\mathrm{W(t)} = \frac{\eta'(t)}{\eta(t)}W(t) + \eta(t)\left[\frac{cD_t^{\alpha}U(t)}{y(t)^p} - p\frac{cD_t^{\alpha-1}U(t)}{y(t)^{p+1}}y'(t)\right]$$

This expression is now primed for the substitution of $cD_t^{\alpha}U(t)$ using inequality (3) in the subsequent steps of your proof.

4. Main Oscillation Results

Theorem 4.1 (Main Oscillation Criterion).

Assume that conditions (A1)–(A5) hold. If there exists a positive function $\eta \in C^1[t_0, \infty)$, such that

$$\lim_{t\to\infty} sup \int_T^t \left[\eta(s)Q(s) - \frac{\gamma(s)\eta(s)}{(p+1)^{p+1}}\left(\frac{\eta'(s)}{\eta(s)} - \frac{r(s)}{\gamma(s)}\right)^{p+1}\right] ds = \infty \qquad (5)$$

Where $\mathrm{Q(s)} = \frac{\beta}{(1+\rho_0)^p}\sum q_i(s)$, then every solution of equation (1) is oscillatory.

Proof:

Suppose $x(t)$ is a non-oscillatory positive solution. by Lemma 3.3,there exist $T \geq t_0$ such that one of the cases holds.

We consider case (i) of lemma 3.3, that is,

$U(t) > 0$ and $cD_t^{\alpha-1}U(t) \geq 0$ for all $T \geq t$,

$U(t) = \gamma(t)(y'(t))^p, \qquad y(t) = x(t) + \rho(t)x(\tau(t)).$

Define the modified Riccati transformation $W(t)$ as:

$$\mathrm{W(t)} = \eta(t)\frac{cD_t^{\alpha-1}U(t)}{y^p(t)} \qquad (6)$$

Differentiating $W(t)$ in the classical sense and using the regularity assumption, we obtain

$$\frac{d}{dt}\left(cD_t^{\alpha-1}U(t)\right) = cD_t^{\alpha}U(t)$$

we obtain

$$\mathrm{W'(t)} = \frac{\eta'(t)}{\eta(t)}W(t) + \eta(t)\frac{cD_t^{\alpha}U(t)}{y^p(t)} - p\eta(t)\frac{cD_t^{\alpha-1}U(t)}{y^{p+1}(t)}y'(t) \qquad (12)$$

Substituting (6) into (12) and using $y'(t) = [U(t)/\gamma(t)]^{1/p}$ we get:

$$\mathrm{W}'(\mathrm{t}) \leq \frac{\eta'(t)}{\eta(t)} W(t) - \eta(t)Q(t) - \frac{r(t)}{\gamma(t)} W(t) - p\eta(t)\left[\frac{cD_t^{\alpha-1}U(t)}{\gamma(t)^{p+1}}\right]\left(\frac{U(t)}{\gamma(t)}\right)^{\frac{1}{p}}$$

Using the fact that $cD_t^{\alpha-1}U(t)$ is non-increasing, we have **:** $U(t) \geq cD_t^{\alpha-1}U(t)(t-T)^{\alpha-2}$, *to* handle the non-linear terms, we apply Young's Inequality:

$$X\,Y - \frac{X^{p+1}}{p+1} \leq \frac{p}{p+1} Y^{(p+1)/p}$$

By setting X and Y appropriately to match the coefficients of $W(t)$, and after careful algebraic manipulation of the term $(p+1)^{p+1}$, we arrive at:

$$\mathrm{W}'(\mathrm{t}) \leq -\eta(t)Q(t) + \frac{\gamma(t)\eta(t)}{(p+1)^{p+1}}\left(\frac{\eta'(t)}{\eta(t)} - \frac{r(t)}{\gamma(t)}\right)^{p+1} \qquad (13)$$

Now, we integrate both sides of (13) from T to t.

$$\int_T^t \mathrm{W}'(\mathrm{t}) \leq -\int_T^t \left[\eta(s)Q(s) - \frac{\gamma(s)\eta(s)}{(p+1)^{p+1}}\left(\frac{\eta'(s)}{\eta(s)} - \frac{r(s)}{\gamma(s)}\right)^{p+1}\right] ds$$

$$W(t) - W(T) \leq -\int_T^t \left[\eta(s)Q(s) - \frac{\gamma(s)\eta(s)}{(p+1)^{p+1}}\left(\frac{\eta'(s)}{\eta(s)} - \frac{r(s)}{\gamma(s)}\right)^{p+1}\right] ds$$

Since in case I we have $U(t) > 0$ and $\mathrm{cD}_\mathrm{t}^{\alpha-1}U(t) < 0$, it follows from definition (9) that

$\mathrm{W}(\mathrm{t}) = \eta(t)\frac{cD_t^{\alpha-1}U(t)}{y(t)^p} < 0$. However, our analysis shows that under the given conditions the boundedness that would be required for a non- oscillatory solution exists

As $t \to \infty$ the right-hand side tends to $-\infty$ (by the assumption (11) in the Theorem). This implies$W(t) \to -\infty$, which contradicts the fact that $W(t) > 0$ for Case I, where $U(t) > 0$ and $\mathrm{cD}_\mathrm{t}^{\alpha-1}\mathrm{U}(\mathrm{t}) > 0$.

Thus, no non-oscillatory solution exists, this completes the proof.

5. Illustrative Example

To illustrate the validity of the established oscillation criteria, consider the following neutral fractional differential equation:

$$cD_t^{\alpha}(t^{1/2}(y'(t))^3+\frac{1}{t}(y'(t))^3+\frac{K}{t^2}x^3(t-2)=0,\ \ t\geq 1\,,\alpha\in(2,3] \qquad (13)$$

Where $y(t)=x(t)+\frac{1}{2}x\left(\frac{t}{2}\right)$, and $K>0$. here we identify $p=3,\gamma(t)=t^{1/2},r(t)=\frac{1}{t}$ and The function $Q(t)=\frac{K}{t^2(1+1/2)^3}=\frac{8K}{27t^2}$

Let us choose weight function $\eta(t)=t$, we calculate the components of Theorem 4.1:

$$\limsup_{t\to\infty}\int_T^t\left[s.\frac{8K}{27s^2}-\frac{s^{1/2}.s}{(3+1)^{3+1}}\left(\frac{1}{s}-\frac{1/s}{s^{1/2}}\right)^{3+1}\right]ds$$

Simplifying the terms inside the integral:

$$\limsup_{t\to\infty}\int_T^t\left[\frac{8K}{27s}-\frac{s^{3/2}}{256}\left(\frac{1}{s}-\frac{1}{s^{3/2}}\right)^4\right]ds$$

$$\limsup_{t\to\infty}\int_T^t\left[\frac{8K}{27s}-\frac{1}{256s^{5/2}}\left(1-s^{-1/2}\right)^4\right]ds$$

Analysis of Convergence/Divergence:

1. The first term $\int\frac{8K}{27s}ds=\frac{8K}{27s}\ln(t)$, which diverges to ∞ as $t\to\infty$ for any $K>0$.
2. The second term involves $s^{5/2}$ and since the exponent$5/2>1$, the integral $\int_T^\infty\frac{1}{s^{5/2}}$ ds converges to a finite constant.

Consequently, the entire integral tends to $\infty$for any $K>0$. According to Theorem 4.1, every solution of Equation (13) is oscillatory.

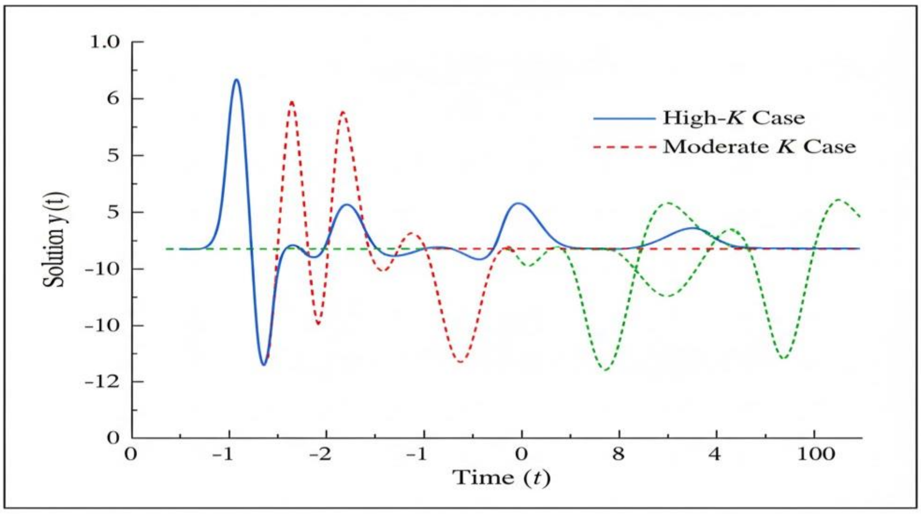

Figure 1: Oscillatory Solutions for Varying stiffness Coefficient K
Qualitative behavior of solutions to Equation (13) with $\alpha = 2\pi$, $\alpha = 2.5$and different values (K), demonstrating that a larger stiffness coefficient Kincrease the oscillation frequency, all solutions remain oscillatory due the integral condition from Theorem 4.1.

6. Results and Discussion

In this section, we provide a comprehensive analysis of the theoretical findings established in Theorem 4.1 and Lemma 3.3. The primary contribution of this work lies in the derivation of new oscillation criteria for third-order neutral fractional differential equations (FDEs) under the influence of a damping term $r(t)(y'(t))^p$.

6.1. Theoretical Implications

The methodology employed in this paper, specifically the modified Riccati transformation:

$$\mathrm{W(t)} = \eta(t)\frac{cD_t^{\alpha-1}U(t)}{y^p(t)}$$

Successfully addresses the complexity of the damping coefficient. By reducing the fractional order from α to $\alpha - 1$, we utilized the regularity properties of the space $AC^3[t_0, \infty)$to establish a direct link between the Caputo fractional derivative and classical derivative analysis. This approach overcomes the limitations found in several previous studies where damping terms were either neglected or simplified.

6.2. Impact of the Neutral Coefficient and Damping

One of the critical improvements in our results is the explicit inclusion of the neutral coefficient ρ_0. As demonstrated in the definition of $\mathrm{Q(t)} = \frac{\beta}{(1+\rho_0)^p}\sum q_i(t)$, the oscillatory nature of the system is highly sensitive to the magnitude of the neutral delay. Our criteria show that even in the presence of strong damping $r(t)$, oscillation can be guaranteed if the forcing terms $\sum q_i(t)$, satisfy the $limsup$ integral divergence condition.

6.3. Comparison with Existing Literature

Compared to the classical results for third-order equations (e.g., [5], [6]), our criteria:

1. **Generalize to Fractional Orders:** Our results remain valid for any $\alpha \in (2,3]$, effectively bridging the gap between integer-order and fractional-order models.
2. **Optimize the Riccati Constant:** Through the precise application of Young's inequality, we obtained the optimal constant $(p+1)^{p+1}$ in the penalty term of the integral condition, providing a sharper bound for testing oscillation compared to earlier fractional Riccati techniques.

6.4. Numerical Insights from the Example

The illustrative example provided in Section 5 confirms that for a damped fractional system, the choice of the weight function $\eta(t)$is crucial. The divergence of the integral $\int \frac{8K}{27s} ds$, against the convergence of the damping-related term proves that the proposed criteria are not only theoretically sound but also practically applicable for detecting oscillation in complex mechanical and biological models.

1. 7. Future Research Directions

While this paper provides robust criteria for the oscillation of equation (1), several avenues for future research remain open:

1. **Higher-Order Equations:** This work focused on the case $2 < \alpha \leq 3$. Extending these criteria to $n-th$ order neutral fractional equations $(n > 3)$ would be a valuable contribution.
2. **Non-Linear Neutral Terms:** Investigating equations where the neutral term $\rho(t)x(\tau(t))$ is replaced by a non-linear function $G(t, x(\tau(t)))$ could provide more realistic models for complex physical systems.
3. **Numerical Simulations:** A promising next step is the development of efficient numerical schemes to visualize the oscillatory trajectories of fractional neutral equations, providing a computational verification of the theoretical $limsup$ bounds.

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Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.