On Certain Generalized Subclasses of Multivalent Functions With Negative Coefficients Defined by a Linear Operator

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حول بعض الأصناف المعممة للدوال المتعددة التكافؤ ذات المعاملات السالبة المعرفة بواسطة مؤثر خطي

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Abstract

In this paper, we investigate several generalized subclasses of multivalent
Analytic functions with negative coefficients defined by a linear operator. Sharp coefficients estimates are obtained and distortion results are established for functions belonging to the proposed classes. Furthermore, we determine the radii o close-to-convexity, starlikeness and convexity associated with these subclasses. Our results extend and unify several earlier works in the theory of multivalent functions and provide new insights into the geometric behavior of functions defined via linear operator.

Keywords: Analytic functions, Multivalent functions, Linear operator, Negative coefficients.

الملخص

تتناول هذه الورقة دراسة بعض الأصناف المعممة من الدوال التحليلية المتعددة التكافؤ ذات المعاملات السالبة والمعرَّفة بواسطة مؤثر خطي. حيث يتم اشتقاق تقديرات حادة لمعاملات هذه الدوال، وإثبات نتائج تتعلق بتشوهها الهندسي ضمن الأصناف المقترحة. كما يتم تحديد أنصاف أقطار التحدب القريب، والنجومية، والتحدب المرتبطة بهذه الأصناف. وتُعد النتائج المتحصل عليها امتدادًا وتوحيدًا لعدد من الأعمال السابقة في نظرية الدوال المتعددة التكافؤ، كما تقدم رؤى جديدة حول السلوك الهندسي للدوال المعرفة بواسطة مؤثرات خطية.

الكلمات المفتاحية: الدوال التحليلية؛ الدوال المتعددة التكافؤ؛ المؤثر الخطي؛ المعاملات السالبة.

1.Introduction

Let $\mathbb{C}$ be the complex plane and $\propto = \{\zeta : \zeta \in \mathbb{C}, |\zeta| < 1\}$ be the open unit disc in $\mathbb{C}$. Also, let $\chi(\propto)$ be the class of holomorphic functions in $\propto$ and $\chi[a,n]$ be the subclass of $\chi(\propto)$ consisting of functions of the form

$$\varphi(\zeta) = a + a_n \zeta^n + a_{n+1} \zeta^{n+1} + \cdots (a \in \mathbb{C}; \ n \in \mathbb{N} = \{1,2,3\})$$

The function $\varphi(\zeta) \in \chi(\propto)$ is said to be $p-$valent in $\propto$ if for each $\sigma \in \propto$, the equation $\varphi(\zeta) = \sigma$ has at most (or exactly) p roots in $\propto$.

Let $\mathcal{H}(p)$ denote the class of all functions the from:

$$\varphi(\zeta) = \zeta^p + \sum_{n=1}^{\infty} a_{n+p} \zeta^{n+p} (p \in \mathbb{N}).$$

Let $T(j,p)$ denote the subclass of $\mathcal{H}(p)$ consisting of all functions of the following from

$$\varphi(\zeta) = \zeta^p - \sum_{k=j+p}^{\infty} a_k \zeta^k \ \ (a_k \geq 0; p,j \in \mathbb{N}). \tag{1.1}$$

Let $(\varphi * g)(\zeta)$ denote the Hadamard product (or convolution) of $\varphi(\zeta)$ and $g(\zeta)$ defined by

$$(\varphi * g)(\zeta) = (g * \varphi)(\zeta) = \zeta^p + \sum_{n=j+p}^{\infty} a_n b_n \zeta^n.$$

Definition 1.1. [4,14]

The Pochhammer symbol or (the shifted factorial), which is denoted by $(\lambda)_k$ is defined (in terms of Gamma function Γ) by

$$(\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \begin{cases} 1 & ; k = 0 \\ \lambda(\lambda + 1)(\lambda + 2) \dots (\lambda + k - 1); & k \in \mathbb{N} \end{cases}$$

Definition 1.2. [19]

For $f \in T(j,p)$, $a,c \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $p \in \mathbb{N}$, the linear operator define as follows:

$$L_{p,j}(a,c)\varphi(\zeta) = \zeta^p - \sum_{k=j+p}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} a_k \zeta^k. \tag{1.2}$$

From (1.2), we deduce that:

$$z(L_{p,j}(a,c)\varphi(\zeta))' = a L_{p,j}(a+1,c)\varphi(\zeta) - (a-p) L_{p,j}(a,c)\varphi(\zeta) \tag{1.3}$$

We also note that:

(i) $L_{p,j}(p+1,p)\varphi(\zeta) = \frac{\zeta \varphi'(\zeta)}{p}$,

(ii) $L_{p,j}(\lambda + p, 1)\varphi(\zeta) = D^{\lambda+p-1}\varphi(\zeta), n \in \mathbb{N}, n > -p$,

Where $D^{\lambda+p-1}$ the generalized Ruscheweyh derivative (see [25]).

(iii) $L_{p,\lambda}(\lambda+p, \lambda+p+1)\varphi(\zeta) = J_{p,\lambda}\varphi(\zeta) = \frac{\lambda+p}{z^\lambda} \int_0^\zeta t^{\lambda-1} \varphi(t)dt, \lambda - p$ where $J_{p,\lambda}$ the generalized Libera integral operator (see [21])

Definition 1.3.[11]

A function $\varphi(\zeta)$ belonging to $T(j,p)$ is in the class $T(j, p, \alpha, \delta)$ iff

$$Re\left(\frac{z\left(L_{p,j}(a,c)f\right)' + \delta z^2\left(L_{p,j}(a,c)f\right)''}{(1-\delta)L_{p,j}(a,c)f + \delta z\left(L_{p,j}(a,c)f\right)'}\right) > \alpha \qquad (1.4)$$

we note that:

   (i)   When $\delta = 0$, we have

$$Re\left(\frac{\zeta\left(L_{p,j}(a,c)\varphi(\zeta)\right)'}{L_{p,j}(a,c)\varphi(\zeta)}\right) > \alpha$$

which is the class of starlike of order α.

   (ii) When $\delta = 0$, $a = p+1$, $c = 1$, we 1have the class

$$Re\left(\frac{z\varphi(\zeta)'}{\varphi(\zeta)}\right) > \alpha; 0 \leq \alpha < P$$

which is the class of starlike functions of order α studied by Owa[30] and Yamakawa [23]

   (iii) When $\delta = 1$, we have

$$Re\left(1 + \frac{\zeta\left(L_{p,j}(a,c)\varphi(\zeta)\right)''}{\left(L_{p,j}(a,c)\varphi(\zeta)\right)'}\right) > \alpha \qquad\qquad 0 \leq \alpha \leq p$$

which is the class of convex operator of order α.

   (iv) When $\delta = 1, a = p+1, c = 1$ we have

$$Re\left(1 + \frac{\zeta\varphi''(\zeta)}{\varphi'(\zeta)}\right) > \alpha; (0 \leq \alpha < p)$$

which is the class of convex functions of order α studied by Owa [26] and Yamakawa [19]

2. Coefficient Estimates.
Theorem 2.1.

Let the function φ defined by (1.1) Then $\varphi \in T(j, p, \alpha, \delta)$ iff

$$\sum_{k=j+p}^\infty \frac{(a)_{k-p}}{(c)_{k-p}}(k-\alpha)[1+\delta(k-1)]a_k \leq (p-\alpha)[1+\delta(p-1)] \qquad (2.1)$$

Proof:

Assume that the inequality (2.1) holds.

Then we have

$$(j + p - \alpha) \sum_{k=j+p}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} [(1 + \delta(k-1))] a_k$$

$$\leq \sum_{k=j+p}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} (k - \alpha)[(1 + \delta(k-1))] a_k \leq (p - \alpha)[1 + \delta(p-1)]$$

that is, that

$$\sum_{k=j+p}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} [(1 + \delta(k-1))] a_k \leq \frac{(p - \alpha)[1 + \delta(p-1)]}{(j + p - \alpha)}$$

$$\left| [1 + \delta(p-1)] - \sum_{k=j+p}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} [(1 + \delta(k-1))] a_k z^{k-p} \right|$$

$$\geq [1 + \delta(p-1)] - \sum_{k=j+p}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} [(1 + \delta(k-1))] a_k |z|^{k-p}$$

$$\geq [1 + \delta(p-1)] + \sum_{k=j+p}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} [(1 + \delta(k-1))] a_k$$

$$\geq \frac{j[1 + \delta(p-1)]}{j + p - \alpha} > 0.$$

Then we find that

$$\left| \frac{\zeta\big(L_{p,j}(a,c)\varphi\big)' + \delta\zeta^2\big(L_{p,j}(a,c)\varphi\big)''}{(1 - \delta)L_{p,j}(a,c)\varphi + \delta\zeta\big(L_{p,j}(a,c)\varphi\big)'} - p \right| \leq p - \alpha$$

this shows that values of the function

$$\varphi(\zeta) =$$

$$\frac{\zeta\big(L_{p,j}(a,c)\varphi\big)' + \delta\zeta^2\big(L_{p,j}(a,c)\varphi\big)''}{(1-\delta)L_{p,j}(a,c)\varphi + \delta\zeta\big(L_{p,j}(a,c)\varphi\big)'} = \frac{\zeta\big(L_{p,j}(a,c)\varphi\big)' + \delta\zeta^2\big(L_{p,j}\varphi\big)''}{(1-\delta)L_{p,j}(a,c)\zeta + \delta\zeta\big(L_{p,j}(a,c)\varphi\big)'} \quad (2.2)$$

then

$$Re\big(\sigma(\zeta)\big) > \alpha \qquad (2.3)$$

if $|\sigma - p| \leq p - \alpha \Rightarrow Re(\sigma) > \alpha$.

For some $\alpha(0 \leq \alpha < p)$, some $\delta(0 \leq \delta \leq 1), p, j \in \mathbb{N}, n \in \mathbb{N}_0$ and $\zeta \in \propto$.
Choose value of ζ on real axis.
So that $\sigma(\zeta)$ given in (2.2) is real. Upon clearing the denominator in (2.3) and letting $\zeta \rightarrow T$ through real value, we can see that

$$p[1 + \delta(p-1)] - \sum_{k=j+p}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} k[1 + \delta(k-1)] a_k$$

$$\geq \left\{ [1 + \delta(p-1)] - \sum_{k=j+p}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} [1 + \delta(k-1)] a_k \right\}$$

thus we have the inequality (2.1).

Corollary 2.1.

Let the function $f(z)$ defined by (1.1) be in the class T (j, p, α, δ), then

$$a_k \leq \frac{(p - \alpha)[1 + \delta(p - 1)]}{\frac{(a)k-p}{(c)k-p}(k - \alpha)[1 + \delta(k - 1)]} \quad , (k \geq p + j, p, j \in \mathbb{N}) \quad (2.4)$$

the result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p - \alpha)[1 + \delta(p - 1)]}{\frac{(a)_{k-p}}{(c)_{k-p}}(k - \alpha)[1 + \delta(k - 1)]} \quad (k \geq j + p, p, j \in \mathbb{N}) \quad (2.5)$$

3. Distortion theorem.

Theorem 3.1.

If a function $f(z)$ defined by (1.1) is in the class $T(j, p, \alpha, \delta)$ then

$$\left\{ \frac{p!}{(p - m)!} - \frac{(p - \alpha)[1 + \delta(p - 1)}{\frac{(a)_j}{(c)_j}(j + p - \alpha)[1 + \delta(j + p - 1)]} |z|^j \right\} |z|^{p-m} \quad (3.1)$$

$$\leq |f^m(z)| \leq \left\{ \frac{p!}{(p-m)!} + \frac{(p-\alpha)[1+\delta(p-1)}{\frac{(a)_j}{(c)_j}(j+p-\alpha)[1+\delta(j+p-1)]} |z|^j \right\} |z|^{p-m}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p - \alpha)[1 + \delta(p - 1)]}{\frac{(a)_j}{(c)_j}(j + p - \alpha)[1 + \delta(j + p - 1)]} \quad (3.2)$$

Proof. In view of Theorem 2.1, we have

$$\frac{\frac{(a)_j}{(c)_j}(j+p-\alpha)[1+\delta(j+p-1)]}{(p-\alpha)[1+\delta(p-1)]} \sum_{k=j+p}^{\infty} k! \, a_k$$

$$\leq \sum_{k=j+p}^{\infty} \frac{\frac{(a)_{k-p}}{(c)_{k-p}}(k - \alpha)[1 + \delta(k - 1)]}{(p - \alpha)[1 + \delta(p - 1)]} a_k \leq 1$$

which readily yields

$$\sum_{k=j+p}^{\infty} k! \, a_k \leq \sum_{k=j+p}^{\infty} \frac{(p - \alpha)[1 + \delta(p - 1)]}{\frac{(a)_j}{(c)_j}(j + p - \alpha)[1 + \delta(j + p - 1)]} a_k \quad (3.3)$$

now, by differentiating both sides of (1.1) m-times, we obtain

$$f^m(z) \leq \frac{p!}{(p - m)!} z^{p-m} - \sum_{k=j+p}^{\infty} \frac{k!}{(k - m)!} a_k z^{k-m}. \quad (3.4)$$

Theorem 3.1, follows from (3.3) and (3.4).

Finally, it is easy to see that the bounds in (3.1) are at tained for function $f(z)$ given by (3.2).

4. Radii of Close –to- convexity, starlikeness and Convexity .

Theorem 4.1. Let the function $f(z)$ defined by (1.1) be in the class

$T(j, p, \alpha, \delta)$ then

i) $f(z)$ is p -valently close-to–convex of order $\varphi(0 \leq \varphi < p) in\ |z| < r_1$, where

$$r_1 = inf_k \left\{ \frac{\frac{(a)k-p}{(c)k-p}(k-\alpha)[1+\delta(k-1)]}{(p-\alpha)[1+\delta(p-1)]} \left(\frac{p-\varphi}{k}\right) \right\}^{\frac{1}{k-p}} , (k \geq j+p, p, j \in \mathbb{N}) \qquad (4.1)$$

ii) $f(z)$ is $p-$ valentlly starlike of order $\emptyset(0 \leq \varphi < p)$ in $|z| < r_2$, where $r_2 =$

$$inf_k \left\{ \frac{\frac{(a)k-p}{(c)k-p}(k-\alpha)[1+\delta(k-1)]}{(p-\alpha)[1+\delta(p-1)]} \left(\frac{p-\varphi}{k-\varphi}\right) \right\}^{\frac{1}{k-p}} , (k \geq j+p, p, j \in \mathbb{N}) \qquad (4.2)$$

iii) $f(z)$ is $p-$valentlly convex of order $\emptyset(0 \leq \varphi < p)$ in $|z| < r_3$, where

$$r_3 = inf_k \left\{ \frac{\frac{(a)k-p}{(c)k-p}(k - \alpha)[1 + \delta(k - 1)]}{(p - \alpha)[1 + \delta(p - 1)]} \cdot \frac{p(p - \varphi)}{k(k - \varphi)} \right\}^{\frac{1}{k-p}} , (k \geq j+p, j, p \in \mathbb{N}) \qquad (4.3)$$

each of these results is sharp for the function $f(z)$ given by (3.2)

Proof. It is sufficient to show that

$$\left| \frac{f'(z)}{z^{p-1}} - \varphi \right| \leq p - \varphi(|z| < r_1; 0 \leq \varphi < p, p \in \mathbb{N}), \qquad (4.4)$$

$$\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \varphi(|z| < r_2; 0 \leq \varphi < p, p \in \mathbb{N}), \qquad (4.5)$$

and

$$\left| 1 + \frac{zf^n(z)}{f'(z)} - p \right| \leq p - \varphi(|z| < r_3 ; 0 \leq \varphi < p, p \in \mathbb{N}), \qquad (4.6)$$

for a function $f(z) \in T(j, p, \alpha, \delta)$ where r_1, r_2 and r_3 are defined by (4.1) – (4.3) respectively. The details involved are fairly straightforward and may omitted.

Conclusions and Recommendations

This part presents some conclusion derived from the conduct of the study of Subordination properties for certain subclasses analytic functions and discued Certain subclasses of multivalent functions with negative coefficients. It also provides some recommendations that can be followed when expanding the stuy.

Conclusions

From our modest study, we conclude the following:
We recommended by study Ruscheweyh operator with p-valent functions.
Catas operator with meromorphic functions and results of this study.
We recommended the study the linear operator $L_{p,j}(a, c)$ associated with meromorphic functions.

Recommendations for future work

- In future work, we recommend studying the same subject, but using other operators not previously studied in the same study.

- Since the study dealt with some certain classes of analytic functions,

other large scale certain classes should be considered

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