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On the Existence and Uniqueness of Fixed Points in 2-Metric Spaces Using Continuity and Cluster Sequences

Abdullah Khalifa Saeed Ali *
Department of Mathematic , Faculty of Science , University of Benghazi. Libya
*Email: aa7697926@gmail.com

حول وجود ووحدانية النقاط الثابتة في الفضاءات المترية الثنائية باستخدام الاستمرارية والمتتاليات المتجمعة

عبدالله خليفه سعيد علي *
قسم الرياضيات، كلية العلوم، جامعة بنغازي، ليبيا

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Abstract

In this paper, we have established a new and significant fixed point result within the framework of 2-metric spaces. The main theorem is formulated under appropriate and carefully structured assumptions that are naturally adapted to the properties structure of 2-metric spaces. In particular, the proof relies essentially on the continuity property of the 2-metric and on the behavior of cluster sequences and their convergence. This continuity of a 2-metric space plays a crucial role in passing to the limit within the maximum - type inequality considered in our result. Furthermore, the concept of cluster sequences and their convergence is employed to guarantee the existence of limit points that satisfy the fixed point condition. The theorem presented here not only ensures the existence of a fixed point but also provides conditions under which the uniqueness of the fixed point can be obtained.

Keywords: Cluster point , Fixed points , 2-metric spaces.

الملخص

في هذا البحث، قمنا بإثبات نتيجة جديدة ومهمة للنقطة الثابتة ضمن إطار الفضاءات المترية ثنائية الأبعاد وقد تمت صياغة النظرية الرئيسية تحت فروض مناسبة ومبنية بعناية، تتلاءم بصورة طبيعية مع البنية والخصائص المميزة الفضاءات المترية ثنائية الأبعاد. وعلى وجه الخصوص، يعتمد البرهان اعتمادًا أساسيًا على خاصية الاستمرارية للمترية الثنائية، وكذلك على سلوك المتتاليات المتجمعة وتقاربها. إذ تلعب خاصية الاستمرارية في الفضاءات المترية ثنائية الأبعاد دورًا جوهريًا في الانتقال إلى النهاية داخل متباينة من نوع القيمة العظمى المعتمدة في نتيجتنا. علاوة على ذلك، تم توظيف مفهوم المتتاليات

المتجمعة وتقاربها لضمان وجود نقاط تحقق شرط النقطة الثابتة. ولا تقتصر النظرية المعروضة هنا على ضمان وجود نقطة ثابتة فحسب، بل تقدم أيضًا شروطًا تكفل وجود نقطة ثابتة وحيدة
الكلمات المفتاحية: نقطة التجميع ، النقاط الثابتة ، الفضاءات المترية ثنائية الأبعاد.

Introduction

The notion of 2-metric spaces was introduced by Gähler in [4] as a natural generalization of the classical concept of metric spaces. The usual metric, which assigns a nonnegative real number to each pair of points, while a 2-metric assigns a nonnegative real number to each ordered triple of points. The theory of 2-metric spaces has attracted considerable attention and has been studied and further developed by many authors. Various fundamental properties of 2-metrics have been investigated, including notions of convergence and continuity adapted to the triple-distance framework. Several researchers have established fixed point theorems in 2-metric spaces under diverse assumptions, generalizing well-known results from standard metric spaces. These contributions have enriched the structure theory of 2-metric spaces and clarified the relationships between metric, 2-metric, and other generalized distance spaces. For examples, see [1], [2], [3], [5], [6], [7] and [8], where important advances in the theory of 2-metric spaces have been presented.

We start by reviewing some essential definitions and preliminary results that will be required in the subsequent discussion.

Definition 1

Let X be a non-empty set and let f be a mapping from X into itself. If there exists an element x in X such that $f(x) = x$, then x is called a *fixed point* of f.

It is evident from induction that if x is a fixed point of f, then x is a fixed point of f^n. That is, $f^n(x) = x$ for all n.

The notion of a 2-metric space, which generalizes the usual metric space concept, is defined as follows :

Definition 2

Let X be a non-empty set and let $d : X \times X \times X \to \mathbb{R}$ be a mapping satisfying the following conditions :

(i) for any two distinct points $x, y \in X$ there exists a point $a \in X$ such that $d(x, y, a) \neq 0$,

(ii) $d(x, y, a) = 0$ if at least two of three points x, y, a are equal,

(iii) $d(x, y, a) = d(x, a, y) = d(y, a, x)$ for all $x, y, a \in X$,

(iv) $d(x, y, a) \leq d(x, y, u) + d(x, u, a) + d(u, y, a)$,

for all $x, y, a, u \in X$.

The function d is called a *2-metric* on X and the pair (X, d) is called a *2-metric space*.

The following is a classical example of a 2-metric space **.**

Example 3 [7]

Let $X = [0,1]$ and define d by

$$d(x, y, a) = x(y - a) + y(a - x) + a(x - y) \quad \text{for all } x, y, a \in X.$$

Then (X, d) is a 2-metric space.

Definition 4

A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be a *convergent sequence* to a point x in X if $\lim_{n \to \infty} d(x_n, x, a) = 0$ for all $a \in X$.

The point x is called the *limit* of the sequence $\{x_n\}$ in X.

This definition is equivalent to :

The sequence $\{x_n\}$ is said to be a *convergent sequence* to x if for each $\varepsilon > 0$, there exists a positive integer N such that

$$d(x_n, x, a) < \varepsilon \quad \text{for all } n > N \text{ and for all } a \in X.$$

If a sequence has a limit in a 2-metric space, then it is unique.

Next, we introduce the concept of a cluster sequence in a 2-metric space.

Definition 5

Let (X, d) be a 2-metric space. The sequence $\{f^n(z)\}$ in X is said to have a *cluster point* u for some z in X if for each $\varepsilon > 0$, there exists a positive integer N such that

$$d(f^n(z), u, a) < \varepsilon \quad \text{for all } n > N \text{ and for all } a \in X.$$

Definition 6

A function f of a 2-metric space (X, d) into itself is called *sequentially continuous at* x in X if for every sequence $\{x_n\}$ in X and $\lim_{n \to \infty} d(x_n, x, a) = 0$ for all $a \in X$, then $\lim_{n \to \infty} d(f(x_n), f(x), a) = 0$.

If a 2-metric is sequentially continuous in two arguments, then it is sequentially continuous in all three arguments.

Definition 7

Let (X, d) be a 2-metric space. A 2-metric function on a space X s called *continuous on* X if it is sequentially continuous in two of its arguments.

In a 2-metric space (X, d), be aware that not all 2-metric functions d are continuous.

Lemma 8 [8]

Let $\{x_n\}$ be a sequence in a 2 - metric space (X, d). If $\{x_n\}$ converges to a point x in X, then $\lim_{n \to \infty} d(x_n, y, a) = d(x, y, a)$ for all $x, y, a \in X$.

Lemma 9 [2]

Let $\{x_n\}$ be a sequence in a 2 - metric space (X, d). If $\{x_n\}$ is bounded below and monotonic decreasing, then the sequence $\{x_n\}$ is convergent.

Main Result

In this section, we present and discuss our result within the framework of 2-metric spaces. The formulation of the theorem is grounded in a thorough analysis of the structural properties inherent to 2-metrics, with particular emphasis on their continuity. Moreover, the proofs and derivations crucially depend on the behavior of cluster sequences, especially their convergence properties, which play a fundamental role in establishing the existence and uniqueness aspects of the result.

Theorem 10

Let (X, d) be a 2-metric space and d be continuous on X. Let f be a continuous mapping of (X, d) into itself satisfying

$$d(f(x), f(y), a) < \max\{d(x, f(x), a), d(y, f(y), a), d(x, y, a)\},$$

for each $x, y, a \in X$, $x \neq y \neq a$.

Suppose that the sequence $\{f^n(z)\}$ has a cluster point u for some z in X.

Then $\{f^n(z)\}$ converges to u and u is a unique fixed point of f.

Proof

Since the sequence $\{f^k(z)\}$ has a cluster point u for some z in X, so

$\lim\limits_{n\to\infty} f^n(z)) = u \quad \rightarrow (1).$

We consider two cases :

Case (i) : Let $f^n(z) = f^{n+1}(z)$ for some non-negative integer n.

Then $f(f^n(z)) = f^{n+1}(z) = f^n(z)$.

Therefore $\lim\limits_{n\to\infty} f(f^n(z)) = \lim\limits_{n\to\infty} f^n(z)$.

Since f is continuous, so

$$f(\lim_{n\to\infty} f^n(z)) = \lim_{n\to\infty} f^n(z).$$

It follows from (1) that $f(u) = u$.

Hence u is a fixed point of f.

For uniqueness : let u and v be two different fixed points of f.

Then $f(u) = u$ and $f(v) = v$.

Therefore $d(u, v, a) = d(f(u), f(v), a)$

$$< \max\{d(u, f(u), a), d(v, f(v), a), d(u, v, a)\}.$$

So $d(u, v, a) < d(u, v, a)$ which is a contradiction.

Thus $u = v$ and hence u is a unique fixed point of f.

Case (ii) : Let $f^{n}(z) \neq f^{n+1}(z)$ for all n.

Define $U : X \to \mathbb{R}$ by

$$U(y) = d(y, f(y), a) \text{ for all } a, y \in X.$$

Then U is a continuous mapping.

We have

$$\begin{aligned} U(f^{n}(z)) &= d(f^{n}(z), f^{n+1}(z), a) \\ &= d(f(f^{n-1}(z)), f(f^{n}(z)), a) \\ &< \max\{d(f^{n-1}(z), f^{n}(z), a),\ d(f^{n}(z), f^{n+1}(z), \\ &\qquad d(f^{n-1}(z), f^{n}(z), a)\} \\ &= \max\{d(f^{n-1}(z), f^{n}(z), a),\ d(f^{n}(z), f^{n+1}(z)\} \\ &= \max\{U(f^{n-1}(z)), U(f^{n}(z))\}. \end{aligned}$$

Therefore $U(f^{n}(z)) < \max\{U(f^{n-1}(z)), U(f^{n}(z))\}$.

If $U(f^{n}(z)) \geq U(f^{n-1}(z))$, then $U(f^{n}(z)) < U(f^{n}(z))$,

which is impossible. Thus

$$U(f^{n}(z)) < U(f^{n-1}(z)),$$

and we can deduce that

$$U(f^{n}(z)) < U(f^{n-1}(z)) < U(f^{n-2}(z)) < U(z) \quad \text{for all } n.$$

Thus the sequence $\{U(f^{n}(z))\}$ is bounded below and monotonic decreasing , so it is a convergent sequence (Lemma 9) .

Let $\lim_{n\to\infty} U(f^{n}(z)) = r$.

$$\begin{aligned} \text{Then} \quad r &= \lim_{n\to\infty} U(f^{n}(z)) \\ &= U(\lim_{n\to\infty} f^{n}(z)) \\ &= U(u), \end{aligned}$$

$$\begin{aligned} \text{and so} \quad U(f(u)) &= U(f(\lim_{n\to\infty} f^{n}(z))) \\ &= U(\lim_{n\to\infty} f^{n+1}(z))) \\ &= \lim_{n\to\infty} U(f^{n+1}(z))) \\ &= r. \end{aligned}$$

We obtain $U(f(u)) = U(u) = r \quad \to (2)$

Now, we will show that u is a fixed point of f. That is, $f(u) = u$.

On contrary, assume that $f(u) \neq u$.

Again, $U(u) = d(u, f(u), a)$ implies that

$$U(f(u)) = d(f(u), f^2(u), a)$$
$$< \max\{d(u, f(u), a), d(f(u), f^2(u), a), d(u, f(u), a)\}$$
$$= \max\{d(u, f(u), a), d(f(u), f^2(u), a)\}$$
$$= \max\{U(u), U(f(u))\}.$$

Then $U(f(u)) < \max\{U(u), U(f(u))\}$.

If $U(u) \leq U(f(u))$, then $U(f(u)) < U(f(u))$ which is impossible.

Thus $U(u) > U(f(u))$ which contradicts (2).

Therefore $f(u) = u$. Thus u is a fixed point of f.

The uniqueness of a fixed point of f is similar to case (i).

Now, $r = \lim_{n\to\infty} U(f^n(z))$

$$= \lim_{n\to\infty} d(f^n(z), f^{n+1}(z), a).$$

By continuity of d, we have

$$r = \lim_{n\to\infty} U(f^n(z))$$
$$= \lim_{n\to\infty} d(f^n(z), f^{n+1}(z), a)$$
$$= d(\lim_{n\to\infty} f^n(z), \lim_{n\to\infty} f^{n+1}(z), a)$$
$$= d(\lim_{n\to\infty} f^n(z), \lim_{n\to\infty} f(f^n(z)), a)$$
$$= d(\lim_{n\to\infty} f^n(z), f(\lim_{n\to\infty} f^n(z)), a)$$
$$= d(u, f(u), a)$$
$$= d(u, u, a)$$
$$= 0.$$

Thus $r = 0$.

we obtain $\lim_{n\to\infty} U(f^n(z)) = r = 0$.

By definition of convergent sequence, for each $\varepsilon > 0$, there exists a positive integer N such that $U(f^k(z)) < \varepsilon$ for all $k > N \quad \to (3)$

Again , since the sequence $\{ f^{k}(z) \}$ has a cluster point u for some z in X , so for each $\varepsilon > 0$, there exists a positive integer N such that

$$d(f^{k}(z), u, a) < \varepsilon \quad \text{for all } k > N \quad \to (4).$$

It follows from (3) and (4) that

$$\max\{U(f^{k}(z)), d(f^{k}(z), u, a)\} < \varepsilon \quad \to (5)$$

Then we have

$$d(f^{n}(z), f^{n}(u), a) = d(f(f^{n-1}(z)), f(f^{n-1}(u)), a),$$
$$< \max\{d(f^{n-1}(z), f^{n}(z), a), d(f^{n-1}(u), f^{n}(u), a), d(f^{n-1}(z), f^{n-1}(u), a)\}.$$

We have already if $f(u) = u$, then $f^{n}(u) = u$ for all n .

Consequently, $d(f^{n-1}(u), f^{n}(u), a) = 0$.

Therefore

$$d(f^{n}(z), f^{n}(u), a) < \max\{d(f^{n-1}(z), f^{n}(z), a), d(f^{n-1}(z), f^{n-1}(u), a)\}$$

$$= \max\{d(f^{n-1}(z), f^{n}(z), a), d(f(f^{n-2}(z)), f(f^{n-2}(u)), a)\}$$

$$< \max\{d(f^{n-1}(z), f^{n}(z), a), \max\{d(f^{n-2}(z), f^{n-1}(z), a), d(f^{n-2}(u), f^{n-1}(u), a), d(f^{n-2}(z), f^{n-2}(u), a)\}\}.$$

Again , since $d(f^{n-2}(u), f^{n-1}(u), a) = 0$, it follows that

$$d(f^{n}(z), f^{n}(u), a) = \max\{d(f^{n-1}(z), f^{n}(z), a), \max\{d(f^{n-2}(z), f^{n-1}(z), a), d(f^{n-2}(z), f^{n-2}(u), a)\}\}$$

$$= \max\{U(f^{n-1}(z)), \max\{U(f^{n-2}(z)), d(f^{n-2}(z), f^{n-2}(u), a)\}\}$$

Since $U(f^{n-1}(z) < U(f^{n-2}(z)$ for all n , so we have

$$d(f^{n}(z), f^{n}(u), a) < \max\{\, U(f^{n-2}(z)), \max\{U(f^{n-2}(z)),$$

$$, d(f^{n-2}(z), f^{n-2}(u), a)\}\}$$

$$= \max\{U(f^{n-2}(z)), d(f^{n-2}(z), f^{n-2}(u), a)\}.$$

Therefore

$$d(f^{n}(z), f^{n}(u), a) < \max\{U(f^{n-2}(z)), d(f^{n-2}(z), f^{n-2}(u), a)\}$$

In the same manner, we obtain for all $n \geq k$,

$$d(f^{n}(z), f^{n}(u), a) < \max\{U(f^{k}(z)), d(f^{k}(z), f^{k}(u), a)\}.$$

Since $f^{n}(u) = u$, it follows that

$$d(f^{n}(z), u, a) < \max\{U(f^{k}(z)), d(f^{k}(z), u, a)\} \rightarrow (6)$$

It follows from (5) and (6) that

$$.d(f^{n}(z), f^{n}(u), a) < \varepsilon$$

Thus the sequence $\{f^{n}(z)\}$ converges to the unique fixed point u .

This completes the proof.

Conclusion

In this work, we have established a new result in the study of fixed points within the framework of 2-metric spaces. The theorem presented provides sufficient conditions ensuring the existence (and potentially uniqueness) of fixed points, extending the understanding of fixed point theory in this generalized setting. This result not only builds upon previous work in the area but also opens the door to further investigations and applications of 2-metric spaces in analysis and related mathematical fields.

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Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.