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# On Probability Distribution of Preference Among Rewards with Their Benefit Function

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# حول التوزيع الاحتمالي للتفضيلات بين المكافآت مع دالة الفائدة

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**Abstract**

assigning a probability function for any reward that a statistician receive is quite challenge. Typically, only he can specify within a certain class of possible distributions, a probability distribution according to his reward will be selected So in this paper we shall specify an appropriate σ-filed F of subset of R (real numbers) and we will consider only probabilities of sets of reward that belongs to F As well as we identify and study the so-called Benefit function of the reward with its property.

**Keywords:** Reward, Benefit function, Probability distribution, σ-filed, Measurability.

**الملخص**

تعيين دالة احتمالية لأي مكافأة يتلقاها الإحصائي يمثل تحديًا كبيرًا. عادةً ما يمكنه فقط تحديد فئة معينة من التوزيعات الاحتمالية، بحيث يتم اختيار توزيع احتمالي وفقًا لمكافأته. لذلك في هذه الورقة سوف نحدد حقل-σ لمجموعة جزئية من الأعداد الحقيقية R وبالتالي سننظر فقط في احتمالات مجموعات المكافأة التي تنتمي إلى حقل-σ كما سنحدد وندرس ما يسمى بدالة الفائدة للمكافأة مع دراسة خواصها.

**الكلمات المفتاحية:** المكافأة، التوزيعات الاحتمالية، دالة المكافأة، حقل-σ، قابلية القياس.

## 1. INTRODUTION

As we mention and defined in [1], statistician probabilities are numerical representations of his beliefs and information, however statistician benefits we defined are numerical representations of individual preferences. So, in this paper we will consider the set R of rewards within the framework elements of Benefit theory. These elements are considered variables; the notation of benefit is used not only in statistics but also in economics and in other field like game theory. Some other examples which may be of interest are: a set of market baskets; a set of unemployment rate or other variables. We assume that the statistician has preferences among rewards in the set *R* and introducing relative likelihoods to inform this framework.

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### 1.1 *Relative Likelihood among rewards*

When comparing two rewards $w_1,\ w_2 \in R$ we express this comparison as:

$w_1 \prec w_2$ to denote that $w_2$ is favoured over $w_1$.
$w_1 \precsim w_2$ to denote that $w_2$ is not favoured to $w_1$.
$w_1 \sim w_2$ to denote that $w_1$ and $w_2$ are equally favoured.

Furthermore, we defined
$w_1 \succ w_2$ to mean that the same as $w_2 \prec w_1$.
$w_1 \succsim w_2$ to mean that the same as $w_2 \precsim w_1$.

The relation $\precsim$ is assumed to possess the following properties:

**1.** For any two rewards $w_1$ and $w_2$ in $R$, then one and only one of the three relations holds

$$w_1 \prec w_2,\ w_1 \succ w_2,\ w_1 \sim w_2$$

**2**. For any three rewards $w_1$ , $w_2$ and $w_3$ in $R$ such that $w_1 \precsim w_2$ and $w_2 \precsim w_3$ then $w_1 \precsim w_3$ .

To avoid a trivial case, we assume that there exists a pair of rewards $s_0$ and $t_0$ in R, where $s_0 \prec t_0,\ s_0 \in$ R , $t_0 \in$ R, indicating that not all rewards are equivalent.

### 1.2 *Relative Likelihood among Preferences Probability Distributions*

Now we consider that $\wp$ is the class of all probability distributions $P$ in $R$, and a set of rewards together with $\sigma$-filed $\mathcal{F}$ of subset of $R$. We extend the notation for preferences from rewards in $R$ to probability distributions in $\wp$. Hence if $P_1 \in \wp$ , $P_2 \in \wp$ any two probability, we have:

$P_1 \prec P_2$ to denote that $P_2$ favoured over $P_1$.
$P_1 \precsim P_2$ to denote that $P_2$ is not favoured over $P_1$.
$P_1 \sim P_2$ to denote that $P_1$ and $P_2$ are equally favoured.

## 2. Result and discussion

***Definition 2.1***

For any two rewards $w_1$ and $w_2$ in $R$ such that $w_1 \precsim w_2$, then the interval $\lfloor w_1,\ w_2 \rfloor$ is a subset of $R$ given by the equation

$$\lfloor w_1\,,\ \ w_2\,\rfloor = \{w\colon\ w_1 \prec w \prec w_2\,\} \in \mathcal{F}\ ,$$

hence for any probability distribution $P \in \wp$ , $P(\lfloor w_1\,,\ \ w_2\,\rfloor)$ is a specific number.

***Definition 2.2***

A probability $P \in \wp$ is considered bounded if rewards exist such that $w_1 \lesssim w_2$, and

$$\text{Prob}(\lfloor w_1, w_2\,\rfloor) = 1,$$

in other words, a distribution $P$ is bounded if it lies within a finite interval $\lfloor r_1\,,\ \ r_2\,\rfloor$.

The relation $\lesssim$ is suppose to have the following two properties:

**1**. If $P_1$ and $P_2$ are any two probability distributions in $\wp_{\text{B}}$, (where $\wp_{\text{B}}$ is the class of all bounded distributions) then exactly one of the three relations must be hold

$$P_1 \prec P_2\ ,\ \ P_1 \succ P_2\ , P_1 \sim P_2\ .$$

**2**. If $P_1$, $P_2$ and $P_3$ are any three distributions in $\wp_{\text{B}}$ such that $P_1 \lesssim P_2$ and $P_2 \lesssim P_3$ then $P_1 \lesssim P_3$ .
Note that, no assumptions have been made so far about preferences for unbounded distributions in $\wp$ .

***Definition 2.3 (definition of the Benefit function)***

Consider a real valued function $f$ defined on the set $R$, and for a probability $P \in \wp$ we denote $E(f|P)$ be the expected value of $f$ (assuming the expectation exist) where $P$ is the probability on $R$, i.e

$$E(f|P) = \int f(r)dP(r)$$

A real $T$ defined on $R$ is called Benefit function if it adhere to the following properties:

Let $P_1 \in \wp$, $P_2 \in \wp$ be any two probability given that both $E(T|P_1)$ and $E(T|P_2)$ exist, then

$$E(T|P_1) \leq\ E(T|P_2)\ .$$

I. For every reward $w \in R$, $T(r)$ is said to be the benefit function of $w$. Hence, one probability will be preferred to another if, and only if the expected benefit of the reward to be received is larger under the first distribution than it under the other distribution.

II. For any probability distribution $P \in \wp$, the number $E(T|P)$ when it exists is often called the benefit of $P$, hence the benefit of probability distribution is equal to the expected benefit of the reward that will be received under that distribution.

The next Lemma proves that if the benefit function exists, then a linear transformation of this function will be also a benefit function.

***Lemma 2.1***

Let $T$ be a benefit function, then any linear transformation of the form $V = aT + b$ $(a, b > 0$ are constants) is also a benefit function.

***Proof***

For any $P \in \wp$, $E(V|P)$, exists if and only if, $E(T|P)$ exists.
Let $P_1 \in \wp$, $P_2 \in \wp$ are any two distributions in which the expectations exist, since $T$ is a benefit function.

$P_1 \lesssim P_2$ if and only, if

$$E(T|P_1) \leq E(T|P_2) \text{ but } E(V|P_i) = E(aT + b|P_i) = aE(T|P_i) + b, \ \ i = 1,2.$$

Since for $a > 0$ hence

$$E(T|P_1) \leq E(T|P_2) \text{ exists if and only, if } E(V|P_1) \leq E(V|P_2),$$

therefore $V$ is a benefit function .

***Example 2.1***

Suppose we have a game dependent upon the outcomes of the random variable $x$ where, $0 \leq x \leq 4$, and we have the benefit function defined by $T(x) = x - 0.5x$, and the variable has the following probability distribution

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P(x)$ | 0.1 | 0.3 | 0.2 | 0.3 | 0.1 |

So we are looking for the expected benefit of its component, so we have

$$E\big(T(x)\big) = E(x - 0.5x) = E(x) - 0.5E(x), \text{ where}$$

$$E(x) = \sum x_i P(x_i) = 0(0.1) + 1(0.3) + 2(0.2) + 3(0.3) + 4(0.1) = 2, \text{ hence}$$

$$E\big(T(x)\big) = 2 - 0.5(2) = 1.$$

### *2.1 The Axiomatic Development of the Benefit function*

From the definition 2.2 it was assumed that the relation $\lesssim$ can be applied to show a simple ordering of the class $\wp_{\mathrm{B}}$ of all bounded probability distributions over the set of rewards in $R$.

In this section we shall begin investigation the further conditions which must be imposed on the relation $\lesssim$, in order exist a function $T$ defined on the set $R$ and which has the following property :

For any two distributions $P_1$ and $P_2$ in the class $\wp_{\mathrm{B}}$, so $P_1 \lesssim P_2$ if and only if,

$$E(T|P_1) \leq E(T|P_2),$$

Namely the function $T$ is benefit function for the distribution in $\wp_{\mathrm{B}}$.

We will use the following in the discussion:

For any two distributions $P_1$ and $P_2$ in the class $\wp_{\mathrm{B}}$, let $\alpha$ be any number with $0 < \alpha < 1$, then the distribution

$$\alpha P_1(B) + (1-\alpha)P_2(B)$$

For every set of rewards $B$ $(B \in \mathcal{F})$ denoted by

$$\alpha P_1 + (1-\alpha)P_2 .$$

Particular if $w_1$ and $w_2$ be any two rewards in $R$, $0 < \alpha < 1$, then

$$\alpha w_1 + (1-\alpha)w_2,$$

denotes the distribution in $\wp$ such that $w_1$ is received with probability $\alpha$ and $w_2$ is received with probability $(1-\alpha)$. Furthermore, it should be noted that if both $P_1$ and $P_2$ are two distributions in $\wp_{\mathrm{B}}$, then the distribution

$$\alpha P_1 + (1-\alpha)P_2$$

also belong to $\wp_{\mathrm{B}}$ .

Now we can introduce to our first assumption:

***Assumption 2.1***

Let $P_1$, $P_2$ and $P$ be any three distributions in the class $\wp_{\mathrm{B}}$, and let $\alpha$ be any number where $0 < \alpha < 1$, then $P_1 \prec P_2$ namely,

$$\alpha P_1 + (1-\alpha)P \prec \alpha P_2 + (1-\alpha)P.$$

Three consequences can be established:

***Lemma 2.2***

let $P_1$ , $P_2$ , $Q_1$ and $Q_2$ are four distributions in the class $\wp_B$, given that $P_1 \prec Q_1$ and $P_2 \prec Q_2$ , and let $\alpha$ be a number where $0 < \alpha < 1$, then

$$\alpha P_1 + (1-\alpha)P_2 \lesssim \alpha Q_1 + (1-\alpha)Q_2.$$

Furthermore if either $P_1 \prec Q_1$ or $P_2 \prec Q_2$ , then

$$\alpha P_1 + (1-\alpha)P_2 \prec \alpha Q_1 + (1-\alpha)Q_2.$$

***Proof***

By two application of

$$\alpha P_1 + (1-\alpha)P_2 \lesssim \alpha Q_1 + (1-\alpha)P_2 \lesssim \alpha Q_1 + (1-\alpha)Q_2. \quad (1)$$

Furthermore if $P_1 \prec Q_1$ or $P_2 \prec Q_2$ , there must be at least one strict preferences in the relation (1).

***Lemma 2. 3***

For any two rewards $w_1$ and $w_2$ in $R$ such that $w_1 \prec w_2$, let $0 < \alpha < 1$, then

$$w_1 \prec \alpha\, w_2 + (1-\alpha)\, w_1 \prec w_2 .$$

***Proof***

Each $w_{\mathrm{i}}$ can be expressed as

$$w_{\mathrm{i}} = \beta\, w_{\mathrm{i}} + (1-\beta)\, w_{\mathrm{i}} ,$$

let $\beta$ be any number, it follows that

$$w_1 = \alpha w_1 + (1-\alpha)w_1 \prec \alpha w_2 + (1-\alpha)w_1 \prec \alpha w_2 + (1-\alpha)w_2 = w_2 . \quad (2)$$

***Lemma 2.4***

For any two rewards $w_1, w_2$ in $R$ such that $w_1 \prec w_2$, and let $\alpha, \beta$ are numbers , $0 \leq \alpha, \beta \leq 1$, then

$$\alpha w_2 + (1 - \alpha) w_1 \prec \beta w_2 + (1 - \beta) w_1 \text{ holds if, and only if, } \alpha < \beta \quad (3)$$

***Proof***

It is enough to proof that if $\alpha < \beta$ thus the relation (3) is true. Assuming that $\alpha < \beta \leq 1$, and let

$$\gamma = (1 - \beta) \div (1 - \alpha) ,$$

hence $0 \leq \gamma < 1$ and it can be verified that

$$\beta w_2 + (1 - \beta) w_1 = \gamma[\alpha w_2 + (1 - \alpha) w_1] + (1 - \gamma) w_2 \quad (4)$$

It follows that

$$\begin{aligned} \alpha w_2 + (1 - \alpha) w_1 &= \gamma[\alpha w_2 + (1 - \alpha) w_1] + (1 - \gamma)[\alpha w_2 + (1 - \alpha) w_1] \\ &\prec \gamma[\alpha w_2 + (1 - \alpha) w_1] + (1 - \gamma) w_2 \\ &= \beta w_2 + (1 - \beta) w_1 . \end{aligned} \quad (5)$$

The next assumption states that a sufficiently slight change in probability will not reverse a strict preference.

***Assumption 2.2***

Let $P_1$ , $P_2$ and $P$ are any three probability distributions in the class $\wp_B$ such that $P_1 \prec P \prec P_2$, then there exist numbers $\alpha, \beta$ , $0 < \alpha, \beta < 1$ such that

$$P \prec \alpha P_2 + (1 - \alpha) P_1 \text{ and } P \succ \beta P_2 + (1 - \beta) P_1 .$$

***Theorem 2.1***

For any rewards $w, w_1$ and $w_2$ in $R$ such that $w_1 \prec w_2$ and $w_1 \lesssim w \lesssim w_2$, there is a unique number $\nu$ such that, $0 \leq \nu \leq 1$ and so

$$w \sim \nu w_2 + (1 - \nu) w_1$$

***Proof***

If $w \sim w_1$ , then $v = 0$ and if $w \sim w_2$ , then $v = 1$, suppose that $w_1 \prec w \prec w_2$, and let $S_1$and $S_2$ be subsets of the unit interval defined as follows:

$$S_1 = \{\alpha: w \prec \alpha w_2 + (1-\alpha)\, w_1 \}$$

$$S_2 = \{\alpha: w \succ \alpha w_2 + (1-\alpha)\, w_1\},$$

then by lemma 2.4, if $\alpha_1 \in S_1$ , and $\alpha_2 > \alpha_1$, then
$\alpha_2 \in S_1$, and if $\alpha_1 \in S_2$, and $\alpha_2 < \alpha_1$, then $\alpha_2 \in S_2$, also (neither $S_1$ nor $S_2 = \emptyset$ ) since
$1 \in S_1$, and $0 \in S_2$.

Therefore $S_1$ is an interval of the form $(\beta, 1]$, and so $S_2$ also an interval of the form $[0, \alpha)$, moreover, it follows from assumption 2.2 that if $\beta_1 \in S_1$, there must be a slightly smaller number $\beta_2$ exist within $S_1$, hence $\beta \notin S_1$ since $S_1$ and $S_2$ are disjoint according with definition.

Then $\alpha \leq \beta$, hence there exist a number $v$ such that , $\alpha \leq v \leq \beta$, since any number $v \notin S_1$, $v \notin S_2$ , it is certain that

$$w \sim v\, w_2 + (1-v) w_1,$$

hence there's always exist a number $v$ satisfying the desired property and from the lemma 2.4 implies that this number is unique .

***2.6 Construction of the Benefit function***

We can now formulate the function $T$ that will act as our benefit function.

Let $s_0$ and $t_0$ are two rewards in $R$ such that $s_0 \prec t_0$, also if $w$ is any reward in $R$ such that $s_0 \precsim w \precsim t_0$, then $T(w)$ is a unique number in the interval , $0 \leq T(w) \leq 1$ that obeys the relation

$$w \sim T(w) t_0 + \big(1 - T(w)\big) s_0, \qquad (6)$$

the existence and uniqueness of the number $T(w)$ are established by theorem 2.1, in particular

we have

$$T(s_0) = 0 \text{ and } T(t_0) = 1,$$

for any $w \in R$ given that $w \prec s_0$, there exist a unique number $\alpha$, $(0 < \alpha < 1)$ that meets the relation

$$s_0 \sim \alpha t_0 + (1-\alpha)w, \quad (7)$$

if the relation (7) is fulfils, we would like the function $T$ to admit the property

$$T(s_0) = \alpha T(t_0) + (1-\alpha)T(w), \quad (8)$$

hence, for

$$T(s_0) = 0 \text{ and } T(t_0) = 1,$$

It follows that

$$T(w) = -\alpha/(1-\alpha)\,. \quad (9)$$

### *2.7 Measurability of the Benefit function*

Finally, we will present here the measurability of the function $T$.

It was assumed by definition 2.1 that if $w_1$ and $w_2$ be any two rewards in $R$ given that $w_1 \lesssim w_2$, then the interval $[w_1, w_2] \in \mathcal{F}$. A stronger assumption will now be introduced:

***Assumption 2.3***

Let $w_1$, $w_2$ and $w_3$ are any three rewards in $R$, and $\alpha, \beta$ are any numbers given that , $0 \leq \alpha, \beta \leq 1$, it must be true that

$$\{w : \alpha w + (1-\alpha)\, w_1 \lesssim \beta w_2 + (1-\beta) w_3\} \in \mathcal{F}\,.$$

***Theorem 2.2***

The function $T$ is a measurable function with regard to $\mathcal{F}$ .

***Proof***

Let $x$ be any real number, so we have

$$\{w : T(w) \leq x\} \in \mathcal{F}, \quad (10)$$

suppose that $x < 0$, if $T(x) \leq x$ for any $w$, then $w \prec s_0$,

thus, it follows from relation (10) and $T(w) = -\alpha/(1-\alpha)$ and lemma 2.4 that $T(w) \leq x$ if, and only if

$$s_0 \gtrsim [-x/(1-x)]t_0 + [1/(1-x)]\,w. \quad (11)$$

But by assumption 2.3, the set of rewards $w$ obeys the relation (11) $\in \mathcal{F}$, hence the relation (10) is correct.

Next suppose that $0 \leq x \leq 1$, then $T(w) \leq x$ if, and only if, $w \lesssim xt_0 + (1-x)s_0$, thus again by assumption 2.3, the relation (10) is correct .

**Conclusion**

The article aims to determine the conditions on the likelihood relations to ensure that the existence of the benefit function of the rewards. On the other hand, we proved that for all classes of the probability distributions in the $\sigma$-filed $\mathcal{F}$, there's exist a set of rewards in $R$ that also belongs to the same $\sigma$-filed $\mathcal{F}$. In addition, we show that our benefit function is a measurable function.

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