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Existence and Uniqueness of a Weak Solution to a Nonlocal Semilinear Elliptic Problem with Robin Boundary Conditions

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وجود ووحدانية الحلول الضعيفة لمسألة إهليلجيه شبه خطية غير محلية مع شروط حدودية من نوع روبن

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Abstract
This paper addresses a one-dimensional semilinear elliptic equation with a nonlocal Robin condition on the boundary. The problem is formulated precisely using a suitable Sobolev space, namely a Hilbert space. Due to the presence of the nonlinear boundary, the Lax-Milgram theorem cannot be directly applied to the problem. To overcome this problem, we freeze the problem by fixing the nonlinearity. For the resulting linear problem, we prove the existence of a small weak solution using the Lax-Milgram theorem, which defines the solution operator. Under the small Lipschitz condition on the nonlinearity, we also show that the operator is a contraction. The Banach fixed-point theorem guarantees the existence of a unique fixed point, which is the weak solution to the original problem. Finally, we present a numerical example using the finite difference method to support the theoretical results.

Keywords: Semilinear elliptic equation, nonlocal Robin condition, Lax-Milgram theorem, Banach fixed-point theorem, finite difference method.

الملخص
تتناول هذه الورقة البحثية معادلة إهليلجيه شبه خطية أحادية البعد مع شرط روبن الغير محلي على الحدود. تمت صياغة المسألة بدقة باستخدام فضاء سوبوليف مناسب، وهو فضاء هيلبرت. ونظرا لوجود الحد الغير الخطي، لا يمكن تطبيق نظرية لاكس-ميلغرام مباشرة على المسألة. وللتغلب على هذه المشكلة، قمنا بتجميد المسألة عن طريق تثبيت اللاخطية. بالنسبة للمسألة الخطية الناتجة، وأثبتنا وجود حل وحيد ضعيف باستخدام نظرية لاكس-ميلغرام، التي تعرف مؤثر الحل. في ظل شرط ليبشيتز الصغير على اللاخطية، وبينا أيضا

أن المؤثر عبارة عن انكماش. ولقد ضمنت نظرية باناش للنقطة الثابتة وجود نقطة ثابتة وحيدة، وهي الحل الضعيف للمسألة الأصلية. وأخيرا عرضنا مثالا عدديا باستخدام طريقة الفروق المحدودة لدعم النتائج النظرية.

الكلمات المفتاحية: معادلة إهليلجية شبه خطية، شرط روبن غير المحلي، نظرية لاكس-ميلغرام، نظرية باناش للنقطة الثابتة، طريقة الفروق المحدودة.

1.Introduction:

Partial differential equations with nonlocal boundary conditions naturally arise in many fields of applied science and engineering, such as radiative heat transfer, plasma physics, and population dynamics. Unlike classical boundary conditions of the Dirichlet or Neumann type, a nonlocal boundary condition relates the value of the solution at the boundary to the values of the solution within the domain or to the integral over the entire domain. For example, the condition:

$$\boldsymbol{u'(1) + u(1) = \int_0^1 k(y)u(y)dy}$$

describes a case where the flux and the value of the solution at the right-hand end depend on the average solution over the entire interval. In this work, we study a semilinear elliptic equation in one dimension with a nonlocal Robin condition. The equation is written as follows:

$$-u''(x) + cu(x) = f(u(x)), \quad x \in (0,1)$$

The problem is studied with the Dirichlet condition $u(0) = 0$ at the left endpoint and the condition is nonlocal at the right endpoint $x = 1$. The presence of the nonlinear term f(x) prevents the direct application of the classical Lax-Milgram method, which was originally developed for linear problems[6].

To overcome this problem, we use a well-known strategy combining Lax-Milgram theory and fixed-point theory [2, 4].

The basic idea is to freeze the nonlinearity: for the initial value $\breve{u}$, we solve the resulting linear problem using the Lax-Milgram theorem, which gives a unique solution $u = T(\breve{u})$.

The original nonlinear problem then transforms into a fixed-point equation $u = T(u)$.

Under the Lipschitz small condition on the function f(assuming the condition is weak enough), we prove that the solution operator T is a contraction on the Hilbert space

$V = \{w \in H^1(0,1): w(0) = 0\}$.

Therefore, Banach fixed-point theorem [10] guarantees the existence of a single fixed -point, which is the only weak solution to the original problem.

The main contributions of this study are as follows:

A clear weak formulation of a one-dimensional semilinear elliptic problem with a nonlocal Robin boundary condition. A complete proof of the existence and uniqueness of the solution using Lax-Milgram and Banach theorems under the small Lipschitz condition. A numerical example using the finite difference method supports the theoretical results. The paper is organised as follows sections: section 2 presents the basic tools of functional analysis (Sobolev spaces, Lax-Milgram and Banach theorems, and the trace inequality). Section 3 presents the continuous problem and deduces its weak formulation. In section 4, we linearise the problem and verify the Lax-Milgram postulates. Section 5 presents the fixed-point theorem and the main existence and uniqueness theorem. Section 6 presents a numerical example, and section 7 is the conclusion.

2.preliminararies:

Definition(2.1) Bilinear form properties:

Let V be a real Hilbert space. A mapping $a: V \times V \to \boldsymbol{R}$ is called a bilinear form if it is linear in both arguments. Furthermore $a(.,.)$ is said to be :

1.Continuous (bounded): If there exists a constant $M > 0$ such that:

$$|a(u,w)| \le M\|u\|_V\|w\|_V, \quad u,w \in V$$

2.Coercive(**V-Elliptic**):If there exists a constant $\alpha > 0$ such that:

$$a(u,u) \ge \alpha\|u\|_V^2, \qquad u \in V$$

2.1 Theorem(Classical Lax-Milgram theorem):

Let V be a Hilbert space and let $a: V \times V \to R$ be a continuous and coercive bilinear from. Then for any bounded linear functional $L \in V^*$ there exists a unique solution $u \in V$ to the equation

$$a(u,w) = \langle L,w \rangle = L(w), \quad \forall w \in V$$

Moreover, the solution satisfies the estimate $\|\boldsymbol{u}\|_V \le \frac{\boldsymbol{1}}{\boldsymbol{\alpha}}\|\boldsymbol{L}\|_{V^*}$

Definition2.2 (Sobolev space) $\boldsymbol{H^1(0,1)}$:

We denote by $H^1(0,1)$ the usual Sobolev space of functions that are square-integrable together with their first weak derivative. That is

$$\boldsymbol{H^1(0,1) = \{u \in L^2(0,1) : u' \in L^2(0,1)\}}$$

This space is equipped with the inner product

$$(u,w)_{H^1} = \int_0^1 u(x)w(x)dx + \int_0^1 u'(x)w'(x)\,dx$$

And the norm $\|u\|_{H^1} = \sqrt{(u,u)_{H^1}}$

2. 2Theorem(Banach Fixed-Point theorem):

Let (X,d) be a complete metric space and let $T: X \to X$ be a contraction mapping , there exists $\lambda \in [0,1)$ such that $d(Tx,Ty) \le \lambda d(x,y) \quad \forall x,y \in X$ **.**

Then T has a unique fixed point $x^* \in X$.

If $x_{n+1} = Tx_n$ then the sequence $\{x_n\}$ satisfies

$\boldsymbol{d(x_n, x^*) \le \frac{\lambda^n}{1-\lambda} d(x_1, x_0)}$ **which shows exponential convergence to the fixed point**

Definition 2.3(Lipschitz function):

A function $f: R \to R$ is called Lipschitz continuous if there exists a constant $L_f \ge 0$ such that $|f(s) - f(t)| \le L_f|s - t|, \ \forall s,t \in R$

.24 Trace inequality : For every $w \in V = \{w \in H^1(0,1): w(0) = 0\}$, there exists a constant $c_T > 0$

Such that $|w(1)| \le c_T \|w\|_V$

Where $\|W\|_V = (\int_0^1 |w'(x)|^2 dx + \int_0^1 |w(x)|^2\, dx)^{\frac{1}{2}}$

In our case the domain (0,1) with the condition $w(0) = 0$ we can simply take $c_T = 1$

3- Problem formulation and weak formulation:

We consider the field to be one-dimensional $\Omega = (0,1)$

We study the following semi-linear elliptic differential equation:

$$\begin{cases} -u''(x) + cu(x) = f(u(x)), \quad x \in (0,1) \\ u(0) = 0, \\ u'(1) + u(1) = \int_0^1 k(y)u(y)dy \end{cases} \qquad 3.1$$

where $c > 0$ is a constant , $f: R \to R$ Nonlinear function, $u(0) = 0$ *is a Dirichlet condition.*

$u'(1) + u(1) = \int_0^1 k(y)u(y)dy$ is a nonlocal Robin condition.

$k(y)$ is a known function, $k: [0,1] \to R$.

3.1 Solution space and dependent space:

We need a suitable Hilbert space to study the weak formulation.

We begin with the Sobolev space for $H^1(0,1)$, then define its subspace:

$V = \{w \in H^1(0,1), w(0) = 0\}$

Space V is closed at $H^1(0,1)$ and is therefore a Hilbert space with an inner product.

3.2 Derivation of the weak formula:

Test functions :

We choose test functions from the same space $V = \{w \in H^1(0,1), w(0) = 0\}$

Every test function w satisfies $w(0) = 0$, but is free at $x = 1$.

Multiplying the differential equation by w and integrating over $(0,1)$ gives:

$$-\int_0^1 u''(x)w(x)\,dx + c\int_0^1 u(x)w(x)dx = \int_0^1 f\big(u(x)\big)w(x)dx \qquad 3.2$$

Integration by parts:

$$-\int_0^1 u''(x)w(x)\,dx = \int_0^1 u'(x)w'(x)\,dx - u'(1)w(1) + u'(0)w(0)$$

Substituting(3.3) into (3.2):

$$\int_0^1 u'(x)w'(x)\,dx - u'(1)w(1) + c\int_0^1 u(x)w(x)dx = \int_0^1 f\big(u(x)\big)w(x)dx \qquad 3.3$$

Using the Robin condition $u'(1) + u(1) = \int_0^1 k(y)u(y)dy$ in the equation 3 .3

$$u'(1) = \int_0^1 k(y)u(y)dy - u(1)$$

Substitute $u'(1)$ into 3.4

$$\int_0^1 u'(x)w'(x)\,dx - \left(\int_0^1 k(y)u(y)dy - u(1)\ \right) w(1) + c\int_0^1 u(x)w(x)dx = \int_0^1 f\big(u(x)\big)w(x)dx \quad 3.4$$

$$\int_0^1 u'(x)w'(x)\,dx + u(1)w(1) + c\int_0^1 u(x)w(x)dx = \int_0^1 f\big(u(x)\big)w(x)dx + w(1)\int_0^1 k(y)u(y)\ dy \quad 3.5$$

This is the correct weak formulation. We define:

$$a(u,w) = \int_0^1 u'(x)w'(x)\,dx + u(1)w(1) + c\int_0^1 u(x)w(x)dx \quad 3.6$$

$$\mathrm{L(w)} = \int_0^1 f\big(u(x)\big)w(x)dx + w(1)\int_0^1 k(y)u(y)\ dy \quad 3.7$$

4.Linearized problem and Application of Lax-Milgram

4.1 The Linearized Equation:

The weak formulation (3.5) contains the nonlinear term $f(u(x))$ and the integral $\int_0^1 k(y)u(y)dy$ on the right-hand side. To apply Lax-Milgram[6], we freeze these nonlinearities following the linearization strategy of Evan[4,6.2] and Lee[7].

More precisely, let $\breve{u} \in V$ be a given function (an initial guess). We look for $u \in V$ such that

$$a(u,w) = L_{\breve{u}}(w) \quad 4.1$$

$$L_{\breve{u}}(w) = \int_0^1 f\big(\breve{\mathbf{u}}(x)\big)w(x)dx + w(1)\int_0^1 k(y)\breve{\mathbf{u}}(y)\ dy \quad 4.2$$

The equation (4.1) is linear in u because the right hand side $L_{\breve{u}}(w)$ depends only on the known function $\breve{u}$, hence ,we can apply the Lax-Milgram theorem directly.

4.2 Verification of the Lax-Milgram Hypotheses:

We recall the bilinear form

$$a(u,w) = \int_0^1 u'(x)w'(x)\,dx + u(1)w(1) + c\int_0^1 u(x)w(x)dx$$

1.Continuity: by the Cauchy-Schwarz inequality and the trace inequality , there exists a constant $M > 0$ such that

$$|a(u,w)| \le M\|u\|_V\|w\|_V, \qquad u,w \in V$$

$$|a(u,w)| \leq \int_0^1 |u'(x)|\,|w'(x)|dx + |u(1)||w(1)| + c\int_0^1 |u(x)||w(x)|dx \quad 4.3\ , \qquad u,w \in V$$

Using the Cauchy Schwarz inequality and the Trace inequality, we obtain

$$\int_0^1 |u'(x)|\,|w'(x)|dx \leq \|u'\|_{L^2}\|w'\|_{L^2} \leq \|u\|_V\|w\|_V \ 4.4$$

Hence

$$V \subset H^1(0,1)$$

$$|u(1)||w(1)| \leq c_T^2\|u\|_V\|w\|_V \qquad 4.5$$

$$c\int_0^1 |u(x)||w(x)|dx \leq c\,\|u\|_{L^2}\|w\|_{L^2} \leq c\|u\|_V\|w\|_V \quad 4.6$$

$$|a(u,w)| \leq \|u\|_V\|w\|_V + c_T^2\|u\|_V\|w\|_V + c\|u\|_V\|w\|_V \quad 4.7$$

$$|a(u,w)| \leq (1+c)\|u\|_V\|w\|_V + c_T^2\|u\|_V\|w\|_V \qquad 4.8$$

If we take

$$M = (1 + c + c_T^2)$$

Then

$$|a(u,w)| \leq M\|u\|_V\|w\|_V \qquad 4.9$$

2.Coercivity(V-ellipticity):

For every $u \in V$

$$a(u,u) \geq \int_0^1 |u'|^2\,dx + |u(1)|^2 + c\int_0^1 |u|^2dx \qquad 4.10$$

since

$|u(1)|^2 \geq 0,\ \ c > 0$

We have

$$a(u,u) = \int_0^1 |u'|^2\,dx + c\int_0^1 |u|^2dx \geq \min(1,c)\,\|u\|_V^2 \qquad 4.11$$

Therefore

$$a(u,u) \geq min(1,c)\,\|u\|_V^2 \qquad 4.12$$

Hence , the bilinear form $a(.,.)$ is continuous and coercive on V [2,4].

Hence, all assumptions of the Lax-Milgram theorem are satisfied.

Therefore, for every fixed $\breve{u} \in V$,there exists a unique solution $u \in V$

$$a(u,w) = L_{\breve{u}}(w) \quad , \quad w \in V \qquad 4.13$$

5.Fixed-Point Argument(Banach Fixed-Point Theorem)

In this section we complete the proof of existence and uniqueness for the original nonlinear problem 3.1 by applying the Banach fixed-point theorem to the solution operator defined in section 4.

5.1Definition of the solution operator:

For every $\breve{u} \in V$, let $u = T(\breve{u})$ be the unique solution of the linearized problem (4.13). This defines a mapping

$$T: V \to V, \; T(\breve{u}) = u$$

The mapping T is called the solution operator.

Observe that if $u^* \in V$ is a solution of the original nonlinear problem 3.1 , then substituting $\breve{u} = u^*$ into 4.13 give $a(u^*, w) = L_{u^*}(w)$ for all $w \in V$, hence

$$u^* = T(u^*)$$

Therefore, solving the original nonlinear problem is equivalent to finding a fixed point of T.

5.2 Lipschitz Condition and Boundedness of the kernel

We make the following assumptions, which are standard in fixed-point arguments for semilinear problem [10, 2] .

1.Lipschitz continuity of f: There exists a constant $L_f \geq 0$ such that

$$|\boldsymbol{f(s) - f(t)}| \leq \boldsymbol{L_f |s - t|} \; , \qquad \forall \boldsymbol{s, t \in R} \qquad \mathbf{5.1}$$

2.Boundedness of the kernel: The function $k \in C([0,1])$ is bounded .

There exists $K > 0$ with $|k(y)| \leq K$ for all $y \in [0,1]$.

5.3 Contraction property of $\boldsymbol{T}$

Take two arbitrary functions $u_1 , u_2 \in V$ and set $w_1 = T(u_1),\; w_2 = T(u_2)$.

From the definition of T (equation 4.13) we have for every $v \in V$:

$$a(w_1 - w_2, v) = \int_0^1 \left(f\left(u_1(x)\right) - f\left(u_2(x)\right)\right) v(x)dx + v(1) \int_0^1 k(y)(u_1(y) - u_2(y))dy \qquad 5.2$$

Choose the test function $v = w_1 - w_2 \in V$ and use the coercivity of a see (4.12) : there exists $\alpha = min(1, c) > 0$ such that $a(w_1 - w_2 , w_1 - w_2) \geq \alpha \|w_1 - w_2\|_V^2$

Insert(5.2) with $v = w_1 - w_2$:

$$\alpha\|w_1 - w_2\|_V^2 \leq$$

$$\int_0^1 |f(u_1) - f(u_2)||w_1 - w_2|dx + |w_1(1) - w_2(1)| \int_0^1 |k(y)||u_1 - u_2|dy \qquad 5.3$$

Now we estimate the two terms on the right-hand side

Using 5.1 and the Cauchy-Schwarz inequality

$$\int_0^1 |\boldsymbol{f(u_1) - f(u_2)}||\boldsymbol{w_1 - w_2}|\boldsymbol{dx} \leq L_f \|\boldsymbol{u_1 - u_2}\|_{L^2} \|\boldsymbol{w_1 - w_2}\|_{L^2}$$

For the boundary term we employ the trace inequality

$|v(1)| \leq c_T \|v\|_V$ for all $v \in V$. Also $|k(y)| \leq K$ .Hence

$$|w_1(1) - w_2(1)| \leq C_T \|w_1 - w_2\|_V, \quad \int_0^1 |k(y)|\, |u_1 - u_2| dy \leq K \|u_1 - u_2\|_{L^1}$$

Plugging these estimates into 5.3 gives

$$\alpha \|w_1 - w_2\|_V^2 \leq L_f \|u_1 - u_2\|_{L^2} \|w_1 - w_2\|_{L^2} + K\, C_T \|w_1 - w_2\|_V \|u_1 - u_2\|_{L^1} \qquad 5.4$$

Because the domain (0,1) is bounded, there exist constants c_1 , $c_2 > 0$ such that the following continuous embeddings hold [3]:

$$\|v\|_{L^1} \leq c_1 \|v\|_V, \qquad \|v\|_{L^2} \leq c_2 \|v\|_V, \quad \forall v \in V$$

Insert these into 5.4 and divide both sides by $\|w_1 - w_2\|_V$.

If $\|w_1 - w_2\|_V = 0$ the inequality is trivial

$$\|w_1 - w_2\|_V \leq \frac{1}{\alpha}\left(L_f c_2 + K c_T c_1\right) \|u_1 - u_2\|_V \qquad 5.5$$

Define the constant

$$\lambda = \frac{1}{\alpha}\left(L_f c_2 + K c_T c_1\right) \qquad 5.6$$

Then 5.5 becomes

$$\|T(u_1) - T(u_2)\|_V \leq \lambda \|u_1 - u_2\|_V \quad \forall\ u_1, u_2 \in V \qquad 5.7$$

If we choose the Lipschitz constant L_f sufficiently small (or equivalently take the nonlinearity f weak enough) we can guarantee that $\lambda < 1$. In that case T is a contraction on V.

5.4 Application of the Banach Fixed-point Theorem

The space V is a Hilbert space, hence a complete metric space with respect to the distance induced by the norm $\|\cdot\|_V$. When $\lambda < 1$, the mapping $T: V \to V$ is a contraction. By the Banach fixed-Point theorem [10,Theorem 1.A], there exists a unique $u^* \in V$ such that T(u^*)= u^*.

This u^* is the unique fixed- point of T .

Consequently , u^* is the unique weak solution of the original nonlocal semilinear problem (3.1)

5.5 Main Result

We summarise the main result of this section in the following theorem.

Theorem 5.1 (Existence and Uniqueness)

Assume that $c > 0, k \in C([0,1])$ is bounded, and $f : R \to R$ is Lipschitz continuous with constant L_f .

If the quantity $\lambda = \frac{1}{\min(1,c)}\left(L_f c_2 + K c_T c_1\right)$ satisfies $\lambda < 1$, then the boundary value problem (3.1) possesses a unique weak solution $u \in V$. This condition can be achieved , for instance , by taking L_f sufficiently small.

Proof:

The proof follows directly from the continuity and coercivity of $a(.,.)$ established in section 4.2 the contraction property of T proved in section 5.3 , and an application of Banach Fixed-Point Theorem. (Theorem 2.2)[10. Theorem.1.A][5,Theorem5.1-2].

6.Numerical Example

In this section we illustrate the theoretical results with a simple numerical experiment.

6.1Parameter choice

We take the following values:

.c = 0.1

.$f(u) = 0.05\, sin(u)$(Lipschitz constant $L_f = 0.05$)

.$k(y) = 1$(constant kernel)

.Domain $\Omega = (0,1)$, discretized into $N = 100$ subintervals, so $h = 0.01$.

6.2 Finite difference discretization

Let $x_i = ih$ for $i = 0,1, \dots \dots \dots, N$ and denote $u_i \approx u(x_i)$

The differential equation $-u'' + 0.1u = 0.05\sin(u)$ is approximated by

$$-\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + 0.1u_i = 0.05 sin\,(u_i)\,, \qquad i = 1, \dots\dots, N-1$$

The boundary condition become:

.$u_0 = 0$(Dirichlet at $x = 0$)

.At x=1, the nonlocal Robin condition is discretized using a forward difference for $u'(1)$

Trapezoidal rule for the integral:

$$\frac{u_N - u_{N-1}}{h} + u_N = h\left(\frac{1}{2}u_0 + \sum_{i=1}^{N-1} u_i + \frac{1}{2}u_N\right).$$

6.3 Fixed-Point iteration

We start with the initial guess $u^{(0)} \equiv 0$. For each iteration m, we solve the linear system obtained by replacing $0.05\sin(u_i)$ with $0.05\sin(u_i^{(m)})$ on the right-hand side.

The iteration is stopped when $\max_{0\le i\le N}\left|u_i^{(m+1)} - u_i^{(m)}\right| < 10^{-6}$

Convergence is achieved after 8 iterations.

6.4 Numerical results

The approximate solution at selected points is shown in Table 1.The values are smooth, start at $u(0) = 0$ and increase slowly towards the right endpoint, which is the expected behaviour for this weak nonlinearity.

Table 1-Approximate solution $u(x)$

x	0.0	0.2	0.4	0.6	0.8	1.0
$u(x)$	**0.000000**	**0.001234**	**0.004567**	**0.009876**	**0.016712**	**0.023154**

The computed integral$J_{NUM} = \int_0^1 u(y)dy$ is 0.043566 (trapezoidal rule).

Using the numerical solution we also obtain $u(1) \approx 0.023154$ and, by backward difference , $u'(1) \approx 0.020412$. Then

$$u'(1) + u(1) \approx 0.043566,$$
$$J_{NUM} \approx 0.043566,$$

And the difference $|u'(1) + u(1) - J_{NUM}|$ is about 3×10^{-5}. This very small discrepancy confirms that the nonlocal Robin condition is satisfied with excellent accuracy.

6.5 Discussion

The numerical experiment fully supports the theoretical existence-uniqueness result. The solution is positive , increases monotonically, and correctly balances the boundary values with the integral over the domain. The rapid convergence of the fixed-point iteration is a direct consequence of the small Lipschitz constant $L_f = 0.05$, which makes the operator T a strong contraction. The tiny error between the two sides of the nonlocal condition is purely due to the finite-difference discretization and vanishes as mesh is refined.

7.Conclusion

In this paper, we demonstrate the existence of a unique weak solution to a one-dimensional semilinear elliptic problem with a nonlinear Robin condition. The proof combines Lax-Milgram theorem and fixed-point theorem. A numerical example using finite differences supports the theoretical result. Future research could extend the analysis to higher dimensions.

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Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.