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Coefficient Estimates for Certain Subclasses of Analytic Functions Defined by the q-Differential Operator

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تقديرات المعاملات لبعض الفئات الجزئية من الدوال التحليلية المعرَّفة بواسطة المؤثر التفاضلي (q)

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Abstract
This paper aims to obtain coefficient estimates for a subclass of normalized analytic functions defined in the open unit disk by means of the q-differential operator. The considered class is characterized by a geometric condition related to a region that is starlike with respect to the point one and symmetric about the real axis. Using concepts from q-calculus and geometric function theory, several coefficient bounds are derived for functions belonging to this class. The obtained results extend and generalize a number of known findings in the theory of analytic and starlike functions. Furthermore, the corresponding results involving the classical derivative are recovered as a special limiting case when the parameter q approaches one from below.

Keywords: Geometric function theory, q-calculus, q-differential operator, analytic function, starlike function, coefficient estimates.

الملخص
يهدف هذا البحث إلى إيجاد تقديرات لمعاملات فئة جزئية من الدوال التحليلية المعيارية المعرَّفة في قرص الوحدة المفتوح باستخدام المؤثر التفاضلي من النوع (q). وتتميز الفئة المدروسة بشرط هندسي يرتبط

بمنطقة نجمية بالنسبة إلى النقطة واحد ومتناظرة حول المحور الحقيقي. وبالاعتماد على مفاهيم حساب (q)ونظرية الدوال الهندسية، تم اشتقاق مجموعة من الحدود والتقديرات لمعاملات الدوال المنتمية إلى هذه الفئة. وتعمل النتائج المتحصل عليها على توسيع وتعميم عدد من النتائج المعروفة في نظرية الدوال التحليلية والدوال النجمية. كما تم الحصول على النتائج المرتبطة بالمشتقة التقليدية بوصفها حالة حدية خاصة عندما يقترب المعامل (q) من الواحد من الجهة اليسرى.

الكلمات المفتاحية: نظرية الدوال الهندسية، حساب(q) ، المؤثر التفاضلي(q) ، الدوال التحليلية، الدوال النجمية، تقديرات المعاملات.

Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (1)$$

which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C}: |z| < 1\}$. Let S be the subclass of $\mathcal{A}$ consisting of functions which are univalent in Δ . If functions $f(z)$ and $g(z)$ are analytic in Δ, we say that the function $f(z)$ is subordinate to $g(z)$ in Δ, and write $f(z) \prec g(z)$, if there exists a Schwarz function w, which is analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$f(z) = g\big(w(z)\big).$$

Moreover, let P denote the class of analytic functions $p(z)$ in Δ such that $p(0) = 1$ and $\Re e\{p(z)\} > 0$. A function $p \in P$ has the Taylor expansion $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. These functions play a vital role in determining the coefficient bounds of various subclasses of analytic functions.

The class $S^*(h)$, consisting of functions $f \in S$ satisfying

$$\frac{zf'(z)}{f(z)} \prec h(z), \quad (z \in \Delta) \qquad (2)$$

and $C(h)$ be the class of function $f \in S$ satisfying

$$1 + \frac{zf''(z)}{f'(z)} \prec h(z), \quad (z \in \Delta) \qquad (3)$$

These classes, introduced by Ma and Minda (1994), have been instrumental in the study of various geometric properties of analytic functions. They have obtained the coefficient bounds for functions belonging to these and related subclasses. A brief history of the coefficient bounds for functions defined by q-differential operators has been mentioned by Srivastava (2013), Mahmood and Sokol (2016), and Mohammed and Darus (2018).

Recently, the application of q-calculus has provided a significant tool in the study of Geometric Function Theory.

Definition 1:

Let $f \in \mathcal{A}$, The q-differential operator denoted by D_q, is defined as follows:

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad (z \neq 0; \quad 0 < q < 1) \qquad (4)$$

and $D_q f(0) = f'(0)$.

To establish our main results, we require the following lemma due to (Ma and Minda 1994).

Lemma 1:

If $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ is an analytic function with a positive real part in Δ, then

$$|c_n| \leq 2, \ \ (n = 1,2,3, \dots).$$

The inequality is sharp for each n, with the equality holding for the function

$$p(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots$$

Furthermore, for any complex number v, we have

$$|c_2 - v c_1^2| \leq 2 \max\{1, |2v - 1|\}.$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p(z)$ is $\frac{1+z^2}{1-z^2}$, or one of it`s rotations.

Main Results

Our main results are stated as follows:

Theorem 1

let the function $f \in \mathcal{A}$ be in the class $S_q^*(h)$. Then, the initial coefficients of f satisfy the inequality:

$$|a_2| \leq \frac{B_1}{[2]_q - 1} \qquad (6)$$

$$|a_3| \leq \frac{B_1}{[3]_q - 1} \max\left\{1, \left|\frac{B_2}{B_1} + \frac{B_1}{[2]_q - 1}\right|\right\} \qquad (7)$$

where $B_1$and B_2 are the coefficients of the function $h(z) = 1 + B_1 z + B_2 z^2 + \cdots$

Proof

let $f \in S_q^*(h)$. Then, by the definition of subordination, there exists an analytic function $w(z)$ in Δ, satisfying $w(0) = 0$ and $|w(z)| < 1$,such that

$$\frac{z\, D_q f(z)}{f(z)} = h\big(w(z)\big) \qquad (8)$$

Define the function $p(z)$ as in lemma 1. Using the relation $w(z) = \frac{p(z)-1}{p(z)+1}$, we have

$$h\big(w(z)\big) = h\left(\frac{p(z) - 1}{p(z) + 1}\right) = h\left(\frac{1}{2}c_1 z + \frac{1}{2}\right)\left(c_2 - \frac{1}{2}c_1^2\right)z^2 + \cdots \qquad (9)$$

By expanding the right-hand side of (9), we obtain

$$h\big(w(z)\big) = 1 + \frac{1}{2}B_1 c_1 z + \left[\frac{1}{2}B_1\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{1}{4}B_2 c_1^2\right]z^2 + \cdots \quad (10)$$

Now, considering the left-hand side of equation (8), we have

$D_q f(z) = z + [2]_q a_2 z^2 + [3]_q a_3 z^3 + \cdots$

By employing the series expansion for $f(z)$ and performing a straightforward calculation for the left-hand side of equation(8), we obtain:

$$1 + \big([2]_q - 1\big)a_2 z + (([3]q - 1)a_3 - ([2]_q - 1)a_2^2)\, z^2 + \cdots \qquad (11)$$

Comparing the coefficients of z and z^2 in equation (10) and (11), it follows that

$$\left([2]_q - 1\right)a_2 = \frac{1}{2}B_1 c_1 \qquad (12)$$

and

$$\left([3]_q - 1\right)a_3 - \left([2]_q - 1\right)a_2^2 = \frac{1}{2}B_1 c_2 - \frac{1}{4}B_1 c_1^2 + \frac{1}{4}B_2 c_1^2 \qquad (13)$$

From (12), we obtain

$$a_2 = \frac{B_1 c_1}{2\left([2]_q - 1\right)} \qquad (14)$$

Substituting (14) into (13) and solving for a_3 we have

$$a_3 = \frac{B_1}{2\left([3]_q - 1\right)}\left[c_2 - \frac{1}{2}\left(1 - \frac{B_2}{B_1} - \frac{B_1}{[2]_q - 1}\right)c_1^2\right] \qquad (15)$$

Finally, by applying lemma 1 to equations (14) and (15), and utilizing the sharp coefficient bound $|c_n| \leq 2$, we obtain the desired results.

$$|a_2| \leq \frac{B_1}{2\left([2]_q - 1\right)}$$

and

$$|a_3| \leq \frac{B_1}{\left([3]_q - 1\right)}\, max\left[1, \left|\frac{B_2}{B_1} + \frac{B_1}{[2]_q - 1}\right|\right]$$

Remark 1

If we let $q \to 1^-$, then $[n]_q \to n$. and the results in theorem 1 reduce to the classical coefficient bounds for the class $S^*(h)$, specifically:

$|a_2| \leq B_1$ and $|a_3| \leq \max\{B_1, |B_2|\}$ (16)

Proof

Given that

$$\lim_{q\to 1^-}[n]_q = \lim_{q\to 1^-}\frac{1 - q^n}{1 - q} = n \qquad (17)$$

Substituting this limit into the bound for $|a_2|$ yields:

$$\lim_{q \to 1^-} \left(\frac{B_1}{[2]_q - 1} \right) = \frac{B_1}{2-1} = B_1 \qquad (18)$$

A similar application of the limit to the bound for $|a_3|$ leads to the classical results established by Ma and Minda.

Remark 2

The bounds in theorem 1 are sharp. Equality holds for the functions f satisfying:

$$\frac{zD_q f(z)}{f(z)} = h(z^{n-1}), \quad (n = 2,3) \qquad (19).$$

Proof

For $n = 2$, let $f(z) = z + a_2 z^2 + \cdots$ satisfy:

$$\frac{zD_q f(z)}{f(z)} = h(z) = 1 + B_1 z + B_2 z^2 + \cdots \qquad (20)$$

Comparing the coefficients of z on both sides immediately yields:

$$\left([2]_q - 1\right) a_2 = B_1 \quad \Rightarrow \; a_2 = \frac{B_1}{[2]_q - 1} \qquad (21)$$

Taking the absolute value gives $|a_2| = \frac{B_1}{[2]_q - 1}$, which shows that the bound for$|a_2|$ is sharp.

For $n = 3$, let $f(z) = z + a_3 z^3 + \cdots$ satisfying:

$$\frac{zD_q f(z)}{f(z)} = h(z^2) = 1 + B_1 z^2 + B_2 z^4 + \cdots \qquad (22)$$

Comparing the coefficients of z^2 on both sides yields :

$$\left([3]_q - 1\right) a_3 = B_1 \quad \Rightarrow \; a_3 = \frac{B_1}{[3]_q - 1} \qquad (23)$$

Taking the absolute value gives $|a_3| = \frac{B_1}{[3]_q - 1}$, which shows that the bound for$|a_3|$ is sharp.

Application to classes Defined by q-fractional Derivatives

For a fixed function $g \in \mathcal{A}$, let $S^*_{q,\gamma}(h)$ denote the class of function $f \in \mathcal{A}$ such that $(\Omega_q^\gamma f) \in S_q^*(h)$. To characterize this class, we utilize the following construction:

Definition 2. (Cf. Owa and Srivastava, 1987) let f be analytic in a simply connected domain containing the origin. The q-fractional derivative of f of order γ is given by:

$$D_{q,z}^{\gamma} f(z) = \frac{1}{\Gamma_q(1-\gamma)} D_q \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\gamma}}\, d\zeta, \qquad (0 \leq \zeta < 1). \qquad (24)$$

Where the principle branch of $(z - q\zeta)^{\gamma}$ is chosen. Building upon this definition and its extensions to q-fractional calculus, the operator $\Omega_q^\gamma: \mathcal{A} \to \mathcal{A}$ is defined as follows:

$$(\Omega_q^\gamma f)(z) = \frac{\Gamma_q(2-\gamma)}{\Gamma_q(2)} z^{\gamma} D_{q,z}^{\gamma} f(z) \quad (\gamma \neq 2,3,4, \dots)$$

Accordingly, the class $S^*_{q,\gamma}(h)$ contains all functions $f \in \mathcal{A}$ for which the image $\Omega_q^\gamma f$ belongs to the class $S^*_{q,\gamma}(h)$.

Theorem 2

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a member of the class$S^*_{q,\gamma}(h)$. Then the following coefficient bounds hold:

$$|a_2| \leq \frac{B_1}{([2]_q - 1)\Psi_q(2,\gamma)}$$

and

$$|a_3| \leq \frac{B_1}{([3]_q - 1)\Psi_q(3,\gamma)} \max\left\{1, \left|\frac{B_2}{B_1} + \frac{B_1 \Psi_q(3,\gamma)}{([2]_q - 1)\Psi_q^2(2,\gamma)}\right|\right\}$$

Where $\Psi_q(n,\gamma) = \frac{\Gamma_q(2-\gamma)\Gamma_q(n+1)}{\Gamma_q 2\ \Gamma_q(n+1-\gamma)}$.

Conclusions

This paper determined the upper bounds for the initial coefficients $|a_2|$ and $|a_3|$ for a newly introduced subclass of normalized analytic function $S_q^*(h)$ defined on

the open unit disc Δ. The properties of these functions are established by substituting the classical derivative with the
q-differential operator. As a special case of this result, coefficient bounds for classes defined via q-fractional derivatives have been obtained too, which generalized several well-known classical estimates when $q \to 1^{-}$

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Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.