Journal of Libyan Academy Bani Walid 2026
مجلة الأكاديمية الليبية بني وليد
e-ISSN: 3104-3860
Volume 2, Issue 4, 2026, Pages: 278-293
Website: https://journals.labjournal.ly/index.php/Jlabw/index

On Inclusion Properties for Certain Subclasses of Analytic Multivalent Functions Involving a New Extended Generalized Derivative Operator

Haneen Ahmed Almasri [1]*, Aisha Ahmed Amer [2]
[1,2] Department of Mathematics, Faculty of Education-Khoms, El Mergib University, Libya.
*Email: haalmasri@elmergib.edu.ly

حول خصائص الاحتواء لفصول جزئية معينة من دوال تحليلية متعددة التكافؤ متضمنة مؤثر تفاضلي معمم موسع جديد

حنين أحمد المصري [1]* ، عائشة أحمد عامر [2]
[2،1] قسم الرياضيات ، كلية التربية ، جامعة المرقب، الخمس، ليبيا

Received: 16-05-2026	**Accepted:** 09-07-2026	**Published**: 23-07-2026

Abstract

The purpose of the present paper, is making use of the principle of subordination between analytic functions, to introduce and systematically study new subclasses of analytic multivalent functions in the open unit disk E , by means of the extended generalized derivative operator $\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n} f(z)$. Also, inclusion properties of these subclasses of functions will be investigated. In addition, we will define the class of p-valent close-to-convex functions of order β and type α , and the class of quasi- convex functions, and will prove their inclusion properties.

Keywords: Analytic p-valent functions, Generalized derivative operator, Inclusion properties, Quasi- convex Open unit disk.

الملخص

الغاية من الورقة الحالية هو استخدام مبدأ التبعية بين الدوال التحليلية، لتعريف ولدراسة منهجية لفصول جزئية جديدة من دوال تحليلية متعددة التكافؤ في قرص الوحدة المفتوحة، باستعمال مؤثر تفاضلي معمم وموسع $\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n} f(z)$ ، كذلك سوف نحقق خصائص الاحتواء لهذه الفصول الجزئية. بالإضافة لذلك، سوف نعرف فصل من دوال قريبة من التحدب من الرتبة β والنوع α ، وفصل من الدوال شبه محدبة، وسنبرهن خواص الاحتواء لها.

الكلمات المفتاحية: خواص الاحتواء، دوال تحليلية متعددة التكافؤ، دوال شبه محدبة، قرص الوحدة المفتوح، مؤثر تفاضلي معمم.

1. Introduction and Preliminaries

Multivalent functions are a natural generalization of univalent functions and constitute an important branch of geometric function theory, a field of significant research interest. One of the most significant studies and applications of such functions was the introduction of new subclasses using generalized derivative operators and studies of their inclusion properties [6,8,21,28]. To establish the theoretical foundation of this topic, this study presents some important concepts in multivalent functions theory.

Consider $\mathcal{A}_p$denote the class of analytic multivalent (p-valent) functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad , \quad (p \in \mathbb{N}), \tag{1}$$

in the open unit disk $E = \{z: z \in \mathbb{C}, |z| < 1\}$. Also,

let $\mathcal{S}_p^*(\alpha)$ denote the class of starlike p-valent functions of order α , that defined by

$$\mathcal{S}_p^*(\alpha) = \left\{f(z) \in \mathcal{A}_p : \mathcal{R}e\left(\frac{zf'(z)}{f(z)}\right) > \alpha, 0 \le \alpha < p, z \in E\right\}, \tag{2}$$

this class introduced by Patil and Thakare [23], in particular $\mathcal{S}_p^*(0) = \mathcal{S}_p^*$the class of starlike p-valent functions in E, which defined by Goodman [15]. The class $\mathcal{C}_p(\alpha)$ of convex p-valent functions of order, is defined by

$$\mathcal{C}_p(\alpha) = \left\{f(z) \in \mathcal{A}_p : \mathcal{R}e\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, 0 \le \alpha < p, z \in E\right\}, \tag{3}$$

this class introduced by Owa [22], in particular $\mathcal{C}_p(0) = \mathcal{C}_p$the class of convex p-valent functions in E, which defined by Goodman [15]. It follows from (2) and (3) that

$$f(z) \in \mathcal{C}_p(\alpha) \quad \text{if and only if} \quad \frac{zf'(z)}{p} \in \mathcal{S}_p^*(\alpha), (0 \le \alpha < p).$$

We denote by $\mathcal{K}_p(\alpha)$ for the class of p-valent close to convex order α , which studies by Mishra and Sahu [20], defined by

$$\mathcal{K}_p(\alpha) = \left\{f \in \mathcal{A}_p : \exists \text{ convex function } g(z), \mathcal{R}e\left(\frac{f'(z)}{g'(z)}\right) > \alpha, 0 \le \alpha < p, z \in E\right\},$$

Notice that, $\mathcal{K}_p(0) = \mathcal{K}_p$ the class of p-valent close to convex function in E.

A function $f \in \mathcal{A}_p$ is said to be p-valent close –to- convex function of order β and type α if there exists a function $g(z) \in \mathcal{S}_p^*(\alpha)$ such that

$$\mathcal{R}e\left(\frac{zf'(z)}{g(z)}\right) > \beta \qquad (0 \le \beta,\ \alpha < p, z \in E), \tag{4}$$

we denote for this class by $R_p(\beta,\alpha)$, the class $R_p(\beta,\alpha)$ was studied by Aouf [7]. We note that, $R_1(\beta,\alpha) = \mathcal{K}_1(\beta,\alpha)$ was studied by Libera [16]. Furthermore, a function $f \in \mathcal{A}_p$ is called quasi-convex of function of order β and type α, if there exists a function $g(z) \in \mathcal{C}_p(\alpha)$ such that

$$\mathcal{R}e\left(\frac{(zf'(z))'}{g'(z)}\right) > \beta \qquad (0 \le \beta,\ \alpha < p, z \in E)\,, \tag{5}$$

we denote for this class by $R_p^*(\beta,\alpha)$. From (4) and (5), clearly

$$f(z) \in R_p^*(\beta,\alpha) \quad \text{if and only if} \quad \frac{zf'(z)}{p} \in R_p(\beta,\alpha). \tag{6}$$

If f and g are analytic functions in E, we say that f is subordinate to g , written $f \prec g$ if there exists a Schwarz function w , which is analytic in E with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in E$such that $f(z) = g(w(z)), z \in E$. Furthermore, if the function g is univalent in E, then (see [10,19])

$$f \prec g \quad \text{if and only if} \quad f(0) = g(0) \quad \text{and} \quad f(E) \subset g(E).$$

Let Q be the class of functions ϕ which are analytic and univalent in E and for which $\phi(E)$is convex with $\phi(0) = 1$and $\mathcal{R}e\{\phi(z)\} > 0, z \in E$. Making use the principle of subordination between analytic functions, we introduce the subclasses $\mathcal{S}_p^*(\alpha;\ \phi)$, $\mathcal{C}_p(\alpha;\phi)$ and $\mathcal{K}_p(\alpha,\gamma;\phi,\psi)$ of the class $\mathcal{A}_p$, $0 \le \alpha,\gamma < p$ $(p \in \mathbb{N})$ and $\phi,\psi \in Q$, which are defined by

$$\mathcal{S}_p^*(\alpha;\phi) = \left\{f(z) \in \mathcal{A}_p:\ \frac{1}{p-\alpha}\left(\frac{zf'(z)}{f(z)} - \alpha\right) \prec \phi(z),\ \ z \in E\right\}, \tag{7}$$

$$\mathcal{C}_p(\alpha;\phi) = \left\{f(z) \in \mathcal{A}_p: \frac{1}{p-\alpha}\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) \prec \phi(z),\ \ z \in E\right\}. \tag{8}$$

And

$$\mathcal{K}_p(\alpha,\gamma;\phi,\psi) = \left\{f \in \mathcal{A}_p: \exists\ \ g \in \mathcal{S}_p^*(\alpha;\phi), s.t\ \frac{1}{p-\gamma}\left(\frac{zf'(z)}{g(z)} - \gamma\right) \prec \psi(z),\ z \in E\right\}.$$

Notice that, from (7) and (8) we have

$$f(z) \in \mathcal{C}_p(\alpha;\phi) \ \text{ if and only if } \ \frac{zf'(z)}{p} \in \ \mathcal{S}_p^*(\alpha;\phi), (0 \le \alpha < p).$$

From these definitions, we can obtain some well-known subclasses by special choices of the functions ϕ and ψ. For example , we have

$$\mathcal{S}_p^*\left(\alpha;\frac{1+z}{1-z}\right) = \mathcal{S}_p^*(\alpha),\quad \mathcal{C}_p\left(\alpha;\frac{1+z}{1-z}\right) = \mathcal{C}_p(\alpha) \quad \text{and} \quad \mathcal{K}_p\left(0{,}0;\frac{1+z}{1-z},\frac{1+z}{1-z}\right) = \mathcal{K}_p.$$

One of the methods used to define generalized differential operators is the concept of convolution, for two functions $f(z)$ given by (1) and $g(z)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \qquad z \in E,$$

the convolution is defined below

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a^k b_k z^k = (g * f)(z).$$

Using the convolution, the authors [2], introduced new extended generalized derivative operator $\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n} f(z)$ for a class $\mathcal{A}_p$ as follows:

let $f(z) \in \mathcal{A}_p$, $m, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, p \in \mathbb{N}$ and $\lambda_2 \geq \lambda_1 \geq 0$, $l \geq 0$. Consider

$$\psi(z) = \frac{p-\lambda_1+l}{p+l}\left(\frac{z}{1-z}\right) + \frac{\lambda_1 z}{p+l}\left(\frac{z}{1-z}\right)',$$

and suppose that

$$\varphi_{p,\lambda_2}^{1}(z) = z^p + \sum_{k=p+1}^{\infty} \frac{1}{1+\lambda_2(k-p)} z^k,$$

then

$$\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n} f(z) = \underbrace{\psi(z) * \quad \cdots \quad * \psi(z)}_{t-times, t=\{0,1,2,\ldots.\}} * \varphi_{p,\lambda_2}^{1}(z) * f(z) * R^{\lambda,p} f(z),$$

Where $R^{\lambda,p} f(z)$ the Rusheweyh derivative operator.

therefore

$$\mathcal{F}_{p,\lambda_1,\lambda_2}^{1,l,n} f(z) = \frac{p-\lambda_1+l}{p+l}\left(\varphi_{p,\lambda_2}^{1}(z) * f\right) + \frac{\lambda_1 z}{p+l}\left(\varphi_{p,\lambda_2}^{1}(z) * f\right)',$$

$$\mathcal{F}_{p,\lambda_1,\lambda_2}^{2,l,n} f(z) = \mathcal{F}_{p,\lambda_1,\lambda_2}^{l,n} (\mathcal{F}_{p,\lambda_1,\lambda_2}^{1,l,n} f(z),$$

$$\vdots$$

$$\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n} f(z) = \mathcal{F}_{p,\lambda_1,\lambda_2}^{1,l,n} (\mathcal{F}_{p,\lambda_1,\lambda_2}^{m-1,l,n} f(z).$$

Thus

$$(9)\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}f(z) = z^p + \sum_{k=p+1}^{\infty}\frac{(p+\lambda_1(k-p)+l)^{m-1}}{(p+l)^{m-1}\left(1+\lambda_2(k-p)\right)^m}\, c^p(n,k)\, a_k z^k,$$

where

$$c^p(n,k) = \binom{n+k-1}{k-p} = \frac{\Gamma(k+n)}{(k-p)!\Gamma(p+n)}.$$

Special cases of this operator includes:

- $\mathcal{F}_{1,\lambda_1,\lambda_2}^{m,l,n}f(z) = I^m(\lambda_1,\lambda_2,l,n)f(z)$,defined by Amer and Darus [1].

- $\mathcal{F}_{1,0,}^{m+1,0,n}f(z) = R^n f(z)$, the Rusheweyh derivative operator [24].

- $\mathcal{F}_{p,1,0}^{m+1,0,0}f(z) = S^n f(z)$, the Salagean derivative operator [26].

- $\mathcal{F}_{1,\lambda_1,0}^{2,0,n}f(z) = R_\lambda^n f(z)$, the generalized Rusheweyh derivative operator [3].

- $\mathcal{F}_{1,\lambda_1,0}^{m+1,0,0}f(z) = S_\beta^n f(z)$, the generalized Salagean derivative operator introduced by Al-oboudi [5].

- $\mathcal{F}_{1,\lambda_1,0}^{m,l,n}f(z) = I^m(\lambda,\beta,l)f(z)$ the Catas derivative operator [11].

- $\mathcal{F}_{1,\lambda_1,\lambda_2}^{m,0,n}f(z) = \mu_{\lambda_1,\lambda_2}^{n,m}f(z)$,defined by AL-Abbad and Darus[4].

- $\mathcal{F}_{p,1,0}^{m+1,0,0}f(z) = S_p^n f(z)$, defined by Eker and Seker [14].

- $\mathcal{F}_{p,0,0}^{m+1,0,n}f(z) = R^{\lambda,p}f(z)$, introduced by Raina and Srivastava [25].

- $\mathcal{F}_{p,0,0}^{m+1,0,m+p-1}f(z) = D^{n+p-1}f(z)$, defined by Kumar and Shukla [27].

- $\mathcal{F}_{p,\lambda_1,0}^{m+1,0,0}f(z) = D_{\delta,p}^n f(z)$,introduced by Bulut [9].

- $\mathcal{F}_{p,\lambda_1,0}^{m,l,0}f(z) = I_p^n(\lambda,l)f(z)$,the catas extended multiplier transformation [12].

It follows from (9) the following relation:

$$(p+l)\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}f(z) = (p+l-\lambda_1)\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}f(z) * \varphi_{p,\lambda_2}^1(z)\right) + \lambda_1 z\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}f(z) * \varphi_{p,\lambda_2}^1(z)\right)' . \quad (10)$$

Now, using the operator $\mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2} f(z)$, we introduce the new subclasses of p-valent analytic $\mathcal{S}^{m,l,n}_{p,\lambda_1,\lambda_2}(\alpha;\phi)$, $\mathcal{C}^{m,l,n}_{p,\lambda_1,\lambda_2}(\alpha;\phi)$, $\mathcal{K}^{m,l,n}_{p,\lambda_1,\lambda_2}(\alpha,\gamma;\phi,\psi)$functions for $\phi,\psi \in \mathcal{Q}$ and $R^{m,l,n}_{p,\lambda_1,\lambda_2}(\beta,\alpha)$, $R^{*m,l,n}_{p,\lambda_1,\lambda_2}(\beta,\alpha)$:

Definition 1.1

For a function $f(z) \in \mathcal{A}_p$, we define the new subclasses of $\mathcal{A}_p$ as

$$\mathcal{S}^{m,l,n}_{p,\lambda_1,\lambda_2}(\alpha;\phi) = \{f(z) \in \mathcal{A}_p:\ \mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2} f(z) \in \mathcal{S}^*_p(\alpha;\phi),\ \ z \in E\}, \tag{11}$$

$$\mathcal{C}^{m,l,n}_{p,\lambda_1,\lambda_2}(\alpha;\phi) = \{f(z) \in \mathcal{A}_p: \mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2} f(z) \in \mathcal{C}_p(\alpha;\phi),\ \ z \in E\}, \tag{12}$$

$$\mathcal{K}^{m,l,n}_{p,\lambda_1,\lambda_2}(\alpha,\gamma;\phi,\psi) = \{f(z) \in \mathcal{A}_p: \mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2} f(z) \in \mathcal{K}_p(\alpha,\gamma;\phi,\psi),\ \ z \in E\}$$

And

$$R^{m,l,n}_{p,\lambda_1,\lambda_2}(\beta,\alpha) = \{f(z) \in \mathcal{A}_p:\ \mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2} f(z) \in R_p(\beta,\alpha),\ \ z \in E\}, \tag{13}$$

$$R^{*m,l,n}_{p,\lambda_1,\lambda_2}(\beta,\alpha) = \{f(z) \in \mathcal{A}_p:\ \mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2} f(z) \in R^*_p(\beta,\alpha),\ \ z \in E\}. \tag{14}$$

Clearly , from (11) and (12) we have

$$f(z) \in \mathcal{C}^{m,l,n}_{p,\lambda_1,\lambda_2}(\alpha;\phi) \quad \text{if and only if} \quad \frac{zf'(z)}{p} \in \mathcal{S}^{m,l,n}_{p,\lambda_1,\lambda_2}(\alpha;\phi).$$

And, from (6) , (13) and (14), we conclude that

$$f(z) \in R^{*m,l,n}_{p,\lambda_1,\lambda_2}(\beta,\alpha) \ \ \text{if and only if} \ \ \frac{zf'(z)}{p} \in R^{m,l,n}_{p,\lambda_1,\lambda_2}(\beta,\alpha). \tag{15}$$

For the purpose of proving our results, we shall make use of the following known results:

Lemma 1.[13] Let ϕ be convex univalent in E with $\phi(0) = 1$and $\mathcal{Re}\{u\phi(z) + v\} > 0$ for $u, v \in \mathbb{C}$. if $p(z)$ is analytic in E with $p(0) = 1$, then

$$,p(z) + \frac{z+p'(z)}{u\,p(z)+v} \prec \phi(z) \quad (z \in E)$$

implies that

$$p(z) \prec \phi(z), \quad (z \in E)$$

(Eenigenbury, Miller, Mocanu& Reade, 1983).

Lemma 2. [19] Let ϕ be convex univalent in E and $w(z)$ be analytic in E with $\mathcal{Re}\{w(z)\} > 0$. if $p(z)$ is analytic in E with $p(0) = \phi(0)$, then

$$p(z) + w(z)\, zp'(z) \prec \phi(z), \quad (z \in E),$$

implies that

$$p(z) \prec \phi(z), \quad (z \in E).$$

(Miller, Mocanu& Reade, 1981).

Lemma 3. [17] let $\varpi(u, v)$ be a complex function $\varpi: D \to \mathbb{C}, \subset \mathbb{C} \times \mathbb{C}$, and let $u = u_1 + iu_2, v = v_1 + iv_2$. Suppose that ϖ satisfies the following conditions:

i. $\varpi(u, v)$is continuous in$D$.

ii. $(1,0) \in D$and $\mathcal{Re}[\varpi(1,0)] > 0$.

iii. $\mathcal{Re}\{\varpi(iu_2, v_1)\} \leq 0$ for all$(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{1}{2}(1 + {u_2}^2)$.

Let $h(z) = 1 + c_1 z + c_2 z + \cdots$ be analytic in E, such that$(h(z), zh(z)) \in D$ for all $z \in E$. If $\mathcal{Re}\{\varpi(h(z), zh'(z))\} > 0,(z \in E)$ then $\mathcal{Re}[h(z)] > 0$ for $z \in E$.

2. Inclusion Properties Involving the Operator $\boldsymbol{\mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2} f(z)}$.

Theorem 1. If $f(z) \in \mathcal{A}_p$and $m, n \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0,\ l \geq 0, z \in E$. Then

$$\mathcal{S}^{m+1,l,n}_{p,\lambda_1,\lambda_2}(\alpha;\phi) \subset \mathcal{S}^{m,l,n}_{p,\lambda_1,\lambda_2}(\alpha;\phi)\,, 0 \leq \alpha < p, p \in \mathbb{N},\ \ \phi \in \mathcal{Q}$$

Proof:

To show that $\mathcal{S}^{m+1,l,n}_{p,\lambda_1,\lambda_2}(\alpha;\phi) \subset \mathcal{S}^{m,l,n}_{p,\lambda_1,\lambda_2}(\alpha;\phi)$, consider $f(z)$ an arbitrary element of the class $\mathcal{S}^{m+1,l,n}_{p,\lambda_1,\lambda_2}(\alpha;\phi)$, It suffices to prove that this element belongs to the class $\mathcal{S}^{m,l,n}_{p,\lambda_1,\lambda_2}(\alpha;\phi)$. Set

$$p(z) = \frac{1}{p-\alpha}\left[\frac{z\left(\mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2} f(z) * \varphi^{1}_{p,\lambda_2}(z)\right)'}{\left(\mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2} f(z) * \varphi^{1}_{p,\lambda_2}(z)\right)} - \alpha\right], \tag{16}$$

where $p(z)$ is analytic function in E, with $p(0) = 1$. Using (10) in (16), we have

$$p(z)=\frac{1}{p-\alpha}\left[\frac{\left(\frac{p+l}{\lambda_1}\right)\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}f(z)-\left(\frac{p+l-\lambda_1}{\lambda_1}\right)\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}f(z)*\varphi_{p,\lambda_2}^{1}(z)\right)'}{\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}f(z)*\varphi_{p,\lambda_2}^{1}(z)\right)}-\alpha\right].$$

We immediately obtain

$$\left(\frac{p+l}{\lambda_1}\right)\frac{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}f(z)}{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}f(z)*\varphi_{p,\lambda_2}^{1}(z)}=p(z)(p-\alpha)+\left(\frac{p+l-\lambda_1}{\lambda_1}\right)+\alpha. \quad (17)$$

Then, differentiating (17) logarithmically with respect to z , we get

Left side

$$log\left\{\left(\frac{p+l}{\lambda_1}\right)\frac{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}f(z)}{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}f(z)*\varphi_{p,\lambda_2}^{1}(z)}\right\}=\frac{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}f(z)*\varphi_{p,\lambda_2}^{1}(z)}{\left(\frac{p+l}{\lambda_1}\right)\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}f(z)}=$$

$$\frac{\left[\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}f(z)*\varphi_{p,\lambda_2}^{1}(z)\right]\left(\frac{p+l}{\lambda_1}\right)\left[\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}f(z)\right]'-\left(\frac{p+l}{\lambda_1}\right)\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}f(z)\left[\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}f(z)*\varphi_{p,\lambda_2}^{1}(z)\right]'}{\left[\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}f(z)*\varphi_{p,\lambda_2}^{1}(z)\right]^2}.$$

Thus

$$log\left\{\left(\frac{p+l}{\lambda_1}\right)\frac{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}f(z)}{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}f(z)*\varphi_{p,\lambda_2}^{1}(z)}\right\}=\frac{\left[\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}f(z)\right]'}{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}f(z)}-\frac{\left[\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}f(z)*\varphi_{p,\lambda_2}^{1}(z)\right]'}{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}f(z)*\varphi_{p,\lambda_2}^{1}(z)},$$

and, right side

$$log\left\{p(z)(p-\alpha)+\left(\frac{p+l-\lambda_1}{\lambda_1}\right)+\alpha\right\}=\frac{p\prime(z)(p-\alpha)}{p(z)(p-\alpha)+\left(\frac{p+l-\lambda_1}{\lambda_1}\right)+\alpha}.$$

Now, by equalizing both sides and multiplying them by z,we have

$$\frac{z\left[\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}f(z)\right]'}{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}f(z)}-\frac{z\left[\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}f(z)*\varphi_{p,\lambda_2}^{1}(z)\right]'}{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}f(z)*\varphi_{p,\lambda_2}^{1}(z)}=\frac{zp'(z)(p-\alpha)}{p(z)(p-\alpha)+\left(\frac{p+l-\lambda_1}{\lambda_1}\right)+\alpha},$$

$$\frac{1}{(p-\alpha)}\left\{\frac{z\left[\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}f(z)\right]'}{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}f(z)}-\frac{z\left[\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}f(z)*\varphi_{p,\lambda_2}^{1}(z)\right]'}{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}f(z)*\varphi_{p,\lambda_2}^{1}(z)}\right\}=\frac{zp\prime(z)}{p(z)(p-\alpha)+\left(\frac{p+l-\lambda_1}{\lambda_1}\right)+\alpha}. \quad (18)$$

Finally, using (16) in (18)we get

$$\frac{1}{(p-\alpha)}\left\{\frac{z\left[\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}f(z)\right]'}{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}f(z)}-\alpha\right\}=p(z)+\frac{zp'(z)}{p(z)(p-\alpha)+\left(\frac{p+l-\lambda_1}{\lambda_1}\right)+\alpha}. \quad (19)$$

Applying lemma 1 to (19), it follows that $p(z) \prec \phi$, that is, $f \in \mathcal{S}_{p,\lambda_1,\lambda_2}^{m,l,n}(\alpha;\phi)$.

Theorem 2. If $f(z) \in \mathcal{A}_p$ and $m,n \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, $l \geq 0$, $z \in E$. Then

$$\mathcal{C}_{p,\lambda_1,\lambda_2}^{m+1,l,n}(\alpha;\phi) \subset \mathcal{C}_{p,\lambda_1,\lambda_2}^{m,l,n}(\alpha;\phi), \quad (0 \leq \alpha < p, p \in \mathbb{N}, \phi \in \mathcal{Q})$$

Proof:

The proof comes in simple steps by utilizing the known relationship between the two classes of convex and starlike functions, as follows:

Since $f(z) \in \mathcal{C}_{p,\lambda_1,\lambda_2}^{m+1,l,n}(\alpha;\phi) \Leftrightarrow \mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n} f(z) \in \mathcal{C}_p(\alpha;\phi)$,

$$\Leftrightarrow \frac{z\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n} f(z)\right)'}{p} \in \mathcal{S}_p^*(\alpha;\phi),$$

$$\Leftrightarrow \mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}\left(\frac{zf'(z)}{p}\right) \in \mathcal{S}_p^*(\alpha;\phi),$$

$$\Leftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}_{p,\lambda_1,\lambda_2}^{m+1,l,n}(\alpha;\phi) \underbrace{\subset \mathcal{S}_{p,\lambda_1,\lambda_2}^{m,l,n}(\alpha;\phi)}_{from\ therom\ 1},$$

$$\Longrightarrow \frac{zf'(z)}{p} \in \mathcal{S}_{p,\lambda_1,\lambda_2}^{m,l,n}(\alpha;\phi)$$

$$\Leftrightarrow \mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}\left(\frac{zf'(z)}{p}\right) \in \mathcal{S}_p^*(\alpha;\phi),$$

$$\Leftrightarrow \frac{z}{p}\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n} f(z)\right)' \in \mathcal{S}_p^*(\alpha;\phi),$$

$$\Leftrightarrow \mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n} f(z) \in \mathcal{C}_p(\alpha;\phi),$$

$$\Leftrightarrow f(z) \in \mathcal{C}_{p,\lambda_1,\lambda_2}^{m,l,n}(\alpha;\phi).$$

The proof is completed.

In theorem 1 and theorem 2 , by taking $\phi(z) = \frac{1+Az}{1-Bz}, -1 \leq B < A \leq 1, z \in E$. We have the following corollary.

Corollary 1. For $(z) \in \mathcal{A}_p$, $\lambda_2 \geq \lambda_1 \geq 0$, $\alpha \geq 0$ and $-1 \leq B < A \leq 1$, we have

$$\mathcal{S}_{p,\lambda_1,\lambda_2}^{m+1,l,n}(\alpha;A,B) \subset \mathcal{S}_{p,\lambda_1,\lambda_2}^{m,l,n}(\alpha;A,B),$$

$$\mathcal{C}_{p,\lambda_1,\lambda_2}^{m+1,l,n}(\alpha; A, B) \subset \mathcal{C}_{p,\lambda_1,\lambda_2}^{m,l,n}(\alpha; A, B).$$

Notice that, when $A = 1, B = -1$, we obtain the classical classes and their related inclusion properties $\mathcal{S}_{p,\lambda_1,\lambda_2}^{m+1,l,n}(\alpha) \subset \mathcal{S}_{p,\lambda_1,\lambda_2}^{m,l,n}(\alpha)$ and $\mathcal{C}_{p,\lambda_1,\lambda_2}^{m+1,l,n}(\alpha) \subset \mathcal{C}_{p,\lambda_1,\lambda_2}^{m,l,n}(\alpha)$.

Theorem 3. If $f(z) \in \mathcal{A}_p$and $m, n \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 > 0$, $l \geq 0$, $z \in E$. Then

$$\mathcal{K}_{p,\lambda_1,\lambda_2}^{m+1,l,n}(\alpha, \gamma; \phi, \psi) \subset \mathcal{K}_{p,\lambda_1,\lambda_2}^{m,l,n}(\alpha, \gamma; \phi, \psi), \ (0 \leq \alpha, \gamma < p\,; p \in \mathbb{N};\ \phi, \psi \in \mathcal{Q})$$

Proof:

Let $f(z) \in \mathcal{K}_{p,\lambda_1,\lambda_2}^{m+1,l,n}(\alpha, \gamma; \phi, \psi) \Rightarrow \exists\ g(z) \in \mathcal{S}_{p,\lambda_1,\lambda_2}^{m+1,l,n}(\alpha; \phi)$, such that

$$\frac{1}{p-\gamma}\left(\frac{z\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n} f(z)\right)'}{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n} g(z)} - \gamma\right) \prec \psi(z), z \in E.$$

Now, put

$$p(z) = \frac{1}{p-\gamma}\left(\frac{z\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n} f(z) * \varphi_{p,\lambda_2}^{1}(z)\right)'}{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n} g(z)} - \gamma\right), \tag{20}$$

where $p(z)$ is analytic in E with $p(0) = 1$. Using (10) in (20), we obtain

$$\left(\frac{p+l}{\lambda_1}\right) \mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n} f(z) = [p(z)(p-\gamma) + \gamma]\, \mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n} g(z) + \left(\frac{p+l-\lambda_1}{\lambda_1}\right)\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n} f(z) * \varphi_{p,\lambda_2}^{1}(z)\right). \tag{21}$$

Differentiating (21) with respect to z and multiplying by z, we get

$$z\left(\frac{p+l}{\lambda_1}\right)\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n} f(z)\right)' = [p(z)(p-\gamma) + \gamma] z \left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n} g(z)\right)' +$$

$$zp'(z)(p-\gamma)\, \mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n} g(z) + z\left(\frac{p+l-\lambda_1}{\lambda_1}\right)\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n} f(z) * \varphi_{p,\lambda_2}^{1}(z)\right). \tag{22}$$

Since $g(z) \in \mathcal{S}_{p,\lambda_1,\lambda_2}^{m+1,l,n}(\alpha; \phi)$, then by theorem 1, we have $g(z) \in \mathcal{S}_{p,\lambda_1,\lambda_2}^{m,l,n}(\alpha; \phi)$, let

$$q(z) = \frac{1}{p-\alpha}\left[\frac{z\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n} g(z)\right)'}{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n} g(z)} - \alpha\right],$$

Where $q(z)$ is analytic function in E, with $q\,(0) = 1$. Using (11) in (21), we have

$$\left(\frac{p+l}{\lambda_1}\right)\frac{\mathcal{F}^{m+1,l,n}_{p,\lambda_1,\lambda_2}g(z)}{\mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2}g(z)} = q(z)(p-\alpha) + \left(\frac{p+l-\lambda_1}{\lambda_1}\right) + \alpha. \tag{23}$$

From (22) and (23), we have

$$\frac{1}{p-\gamma}\left(\frac{z\left(\mathcal{F}^{m+1,l,n}_{p,\lambda_1,\lambda_2}f(z)\right)'}{\mathcal{F}^{m+1,l,n}_{p,\lambda_1,\lambda_2}g(z)} - \gamma\right) = p(z) + \frac{zp'(z)}{q(z)(p-\alpha)+\left(\frac{p+l-\lambda_1}{\lambda_1}\right)+\alpha}.$$

Since $0 \leq \alpha < p$ ($p \in \mathbb{N}$) and $q \prec \phi$ in E, then

$$Re\left\{q(z)(p-\alpha) + \left(\frac{p+l-\lambda_1}{\lambda_1}\right) + \alpha\right\} > 0, \quad z \in E.$$

Hence applying Lemma 2, we can show that $p \prec \psi$, that is, $f \in \mathcal{K}^{m,l,n}_{p,\lambda_1,\lambda_2}(\alpha,\gamma;\phi,\psi)$.

Theorem 4. If $f(z) \in \mathcal{A}_p$ and $m,n \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, $l \geq 0$, $z \in E$. Then

$$R^{m+1,l,n}_{p,\lambda_1,\lambda_2}(\beta,\alpha) \subset R^{m,l,n}_{p,\lambda_1,\lambda_2}(\beta,\alpha)$$

where $(0 \leq \beta,\ \alpha < p, p \in \mathbb{N})$.

Proof:

Let $f(z) \in R^{m+1,l,n}_{p,\lambda_1,\lambda_2}(\beta,\alpha)$. Then, there exists a function $k(z) \in \mathcal{S}^*_p(\alpha)$ such that

$$\mathcal{R}e\left(\frac{z\left(\mathcal{F}^{m+1,l,n}_{p,\lambda_1,\lambda_2}f(z)\right)'}{g(z)}\right) > \beta \quad , \qquad (z \in E).$$

$\mathcal{F}^{m+1,l,n}_{p,\lambda_1,\lambda_2}k(z) = g(z)$,

$$\mathcal{R}e\left(\frac{z\left(\mathcal{F}^{m+1,l,n}_{p,\lambda_1,\lambda_2}f(z)\right)'}{\mathcal{F}^{m+1,l,n}_{p,\lambda_1,\lambda_2}k(z)}\right) > \beta \ , \qquad (z \in E).$$

Now, put

$$\frac{z\left(\mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2}f(z) * \varphi^1_{p,\lambda_2}(z)\right)'}{\mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2}k(z) * \varphi^1_{p,\lambda_2}(z)} = \beta + (p-\beta)h(z),$$

$$0 \leq \beta < 1, z \in E , \tag{24}$$

where $h(z) = 1 + c_1 z + c_2 z + \cdots$. Using (10) we can write

$$\frac{z\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}f(z)\right)'}{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}k(z)}=\frac{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}\left(zf'(z)\right)}{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}k(z)}=$$

$$\frac{\left(\frac{p+l-\lambda_1}{p+l}\right)\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}(zf'(z))*\varphi_{p,\lambda_2}^1(z)\right)+\left(\frac{\lambda_1 z}{p+l}\right)\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}(zf'(z))*\varphi_{p,\lambda_2}^1(z)\right)'}{\left(\frac{p+l-\lambda_1}{p+l}\right)\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}k(z)*\varphi_{p,\lambda_2}^1(z)\right)+\left(\frac{\lambda_1 z}{p+l}\right)\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}k(z)*\varphi_{p,\lambda_2}^1(z)\right)'},$$

$$\Rightarrow\frac{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}(zf'(z))}{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}k(z)}=\frac{\left(\frac{p+l-\lambda_1}{p+l}\right)\frac{\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}(zf'(z))*\varphi_{p,\lambda_2}^1(z)\right)}{\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}k(z)*\varphi_{p,\lambda_2}^1(z)\right)}+\left(\frac{\lambda_1 z}{p+l}\right)\frac{\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}(zf'(z))*\varphi_{p,\lambda_2}^1(z)\right)'}{\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}k(z)*\varphi_{p,\lambda_2}^1(z)\right)}}{\left(\frac{\lambda_1 z}{p+l}\right)\frac{z\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}k(z)*\varphi_{p,\lambda_2}^1(z)\right)'}{\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}k(z)*\varphi_{p,\lambda_2}^1(z)\right)}+\left(\frac{p+l-\lambda_1}{p+l}\right)}. \tag{25}$$

Since $k(z)\in\mathcal{S}_{p,\lambda_1,\lambda_2}^{m+1,l,n}(\alpha)\subset\mathcal{S}_{p,\lambda_1,\lambda_2}^{m,l,n}(\alpha)$, we set

$$\frac{z\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}k(z)*\varphi_{p,\lambda_2}^1(z)\right)'}{\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}k(z)*\varphi_{p,\lambda_2}^1(z)\right)}=\alpha+(p-\alpha)H(z), \tag{26}$$

where $\mathcal{R}e\,H(z)>0,\ (z\in E)$. Using (24) and (26), we can write (25) as

$$\frac{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}(zf'(z))}{\mathcal{F}_{p,\lambda_1,\lambda_2}^{m+1,l,n}k(z)}=\frac{\left(\frac{\lambda_1}{p+l}\right)\frac{z\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}(zf'(z))*\varphi_{p,\lambda_2}^1(z)\right)'}{\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}k(z)*\varphi_{p,\lambda_2}^1(z)\right)}+\left(\frac{p+l-\lambda_1}{p+l}\right)[\beta+(p-\beta)h(z)]}{\frac{\lambda_1}{p+l}[\alpha+(p-\alpha)H(z)]+\left(\frac{p+l-\lambda_1}{p+l}\right)}. \tag{27}$$

On the other hand , from (24) we have

$$z\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}f(z)*\varphi_{p,\lambda_2}^1(z)\right)'=\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}k(z)*\varphi_{p,\lambda_2}^1(z)\right)[\beta+(p-\beta)h(z)]. \tag{28}$$

Differentiating (28), and multiplying by z, we get

$$z\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}\left(f'(z)\right)*\varphi_{p,\lambda_2}^1(z)\right)'=$$

$$z\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}k(z)*\varphi_{p,\lambda_2}^1(z)\right)[(p-\beta)h'(z)]+z[\beta+(p-\beta)h(z)]\left(\mathcal{F}_{p,\lambda_1,\lambda_2}^{m,l,n}k(z)*\varphi_{p,\lambda_2}^1(z)\right)',$$

$$\frac{z\left(\mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2}(zf'(z))*\varphi^{1}_{p,\lambda_2}(z)\right)'}{\left(\mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2}k(z)*\varphi^{1}_{p,\lambda_2}(z)\right)} = z[(p-\beta)h'(z)] + \frac{z\left(\mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2}k(z)*\varphi^{1}_{p,\lambda_2}(z)\right)'}{\left(\mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2}k(z)*\varphi^{1}_{p,\lambda_2}(z)\right)}[\beta+(p-\beta)h(z)]. \tag{29}$$

Now, using (26) in (29), we get

$$\frac{z\left(\mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2}(zf'(z))*\varphi^{1}_{p,\lambda_2}(z)\right)'}{\left(\mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2}k(z)*\varphi^{1}_{p,\lambda_2}(z)\right)} = z[(p-\beta)h'(z)] + [\alpha+(p-\alpha)H(z)][\beta+(p-\beta)h(z)] \tag{30}$$

Consequently, from (30) we can write (27) as

$$\frac{\mathcal{F}^{m+1,l,n}_{p,\lambda_1,\lambda_2}(zf'(z))}{\mathcal{F}^{m+1,l,n}_{p,\lambda_1,\lambda_2}k(z)} = \frac{[z[(p-\beta)h'(z)]+[\alpha+(p-\alpha)H(z)][\beta+(p-\beta)h(z)]]+\left(\frac{p+l-\lambda_1}{\lambda_1}\right)[\beta+(p-\beta)h(z)]}{[\alpha+(p-\alpha)H(z)]+\left(\frac{p+l-\lambda_1}{\lambda_1}\right)},$$

$$\Rightarrow \quad \frac{z\,\mathcal{F}^{m+1,l,n}_{p,\lambda_1,\lambda_2}(zf'(z))}{\mathcal{F}^{m+1,l,n}_{p,\lambda_1,\lambda_2}k(z)} - \beta = (p-\beta)h(z) + \frac{z[(p-\beta)h'(z)]}{[\alpha+(p-\alpha)H(z)]+\left(\frac{p+l-\lambda_1}{\lambda_1}\right)}. \tag{31}$$

Taking $h(z)=u$ and $zh'(z)=v$ in(31), we define the function $\varpi(u,v)$by

$$\varpi(u,v) = (p-\beta)u + \frac{(p-\beta)v}{[\alpha+(p-\alpha)H(z)]+\left(\frac{p+l-\lambda_1}{\lambda_1}\right)},$$

this implies

i. $\varpi(u,v)$is continuous in$D=\mathbb{C}\times\mathbb{C}$.

ii. $(1,0)\in D$and $\mathcal{Re}[\varpi(1,0)]>0$.

To verify condition (iii) in lemma 3., we proceed as follows

$$\mathcal{Re}\{\varpi(iu_2,v_1)\} = \mathcal{Re}\left[(p-\beta)iu_2 + \frac{(p-\beta)v_1}{\left[\alpha+\left(\frac{p+l-\lambda_1}{\lambda_1}\right)\right]+(p-\alpha)h_1(x,y)+i(p-\alpha)h_2(x,y)}\right],$$

$$= \mathcal{Re}\left[(p-\beta)iu_2 + \frac{\left[\left[\alpha+\left(\frac{p+l-\lambda_1}{\lambda_1}\right)\right]+(p-\alpha)h_1(x,y)-i(p-\alpha)h_2(x,y)\right](p-\beta)v_1}{\left[\alpha+\left(\frac{p+l-\lambda_1}{\lambda_1}\right)+(p-\alpha)h_1(x,y)\right]^2+[(p-\alpha)h_2(x,y)]^2}\right],$$

$$\Rightarrow \mathcal{Re}\{\varpi(iu_2, v_1)\} = \frac{\left[\alpha+\left(\frac{p+l-\lambda_1}{\lambda_1}\right)+(p-\alpha)h_1(x,y)\right](p-\beta)v_1}{\left[\alpha+\left(\frac{p+l-\lambda_1}{\lambda_1}\right)+(p-\alpha)h_1(x,y)\right]^2+[(p-\alpha)h_2(x,y)]^2}\ .$$

Where $H(z) = h_1(x,y) + ih_2(x,y)$ and $h_1(x,y)$, $h_2(x,y)$ being the functions of x, y and $\mathcal{Re}\ H(z) = h_1(x,y) > 0$.

By putting $v_1 \leq -\frac{1}{2}(1 + u_2{}^2)$, we have

$$\mathcal{Re}\{\varpi(iu_2, v_1)\} = -\frac{\left[\alpha+\left(\frac{p+l-\lambda_1}{\lambda_1}\right)+(p-\alpha)h_1(x,y)\right](p-\beta)(1+u_2{}^2)}{2\left[\alpha+\left(\frac{p+l-\lambda_1}{\lambda_1}\right)+(p-\alpha)h_1(x,y)\right]^2+[(p-\alpha)h_2(x,y)]^2} < 0.$$

Hence, $\mathcal{Re}\ h(z) > 0$ ($z \in E$) and $f(z) \in R^{m,l,n}_{p,\lambda_1,\lambda_2}(\beta,\alpha)$.

Theorem 5. If $f(z) \in \mathcal{A}_p$ and $m, n \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, $l \geq 0$, $z \in E$. Then

$$R^{*}{}^{m+1,l,n}_{p,\lambda_1,\lambda_2}(\beta,\alpha) \subset R^{*}{}^{m,l,n}_{p,\lambda_1,\lambda_2}(\beta,\alpha), (0 \leq \beta,\ \alpha < p, p \in \mathbb{N})$$

Proof:

The proof comes from the relationship between the classes $R_p(\beta,\alpha)$ and $R_p^*(\beta,\alpha)$ that given by (6) and (15). Then, using Theorem .4, as follows

let $f(z) \in R^{*}{}^{m+1,l,n}_{p,\lambda_1,\lambda_2}(\beta,\alpha) \Leftrightarrow \mathcal{F}^{m+1,l,n}_{p,\lambda_1,\lambda_2} f(z) \in R_p^*(\beta,\alpha)$,

$$\Leftrightarrow \frac{z\left(\mathcal{F}^{m+1,l,n}_{p,\lambda_1,\lambda_2} f(z)\right)'}{p} \in R_p(\beta,\alpha)\ ,$$

$$\Leftrightarrow \mathcal{F}^{m+1,l,n}_{p,\lambda_1,\lambda_2}\left(\frac{zf'(z)}{p}\right) \in R_p(\beta,\alpha)\ ,$$

$$\Leftrightarrow \frac{zf'(z)}{p} \in R^{m+1,l,n}_{p,\lambda_1,\lambda_2}(\beta,\alpha)\ ,$$

$$\Longrightarrow \frac{zf'(z)}{p} \in R^{m,l,n}_{p,\lambda_1,\lambda_2}(\beta,\alpha)\ ,$$

$$\Leftrightarrow \mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2}\left(\frac{zf'(z)}{p}\right) \in R_p(\beta,\alpha)\ ,$$

$$\Leftrightarrow \frac{z}{p}\left(\mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2} f(z)\right)' \in R_p(\beta,\alpha)\ ,$$

$$\Leftrightarrow \mathcal{F}^{m,l,n}_{p,\lambda_1,\lambda_2} f(z) \in R_p^*(\beta,\alpha),$$

$$\Leftrightarrow f(z) \in R^{*m,l,n}_{p,\lambda_1,\lambda_2}(\beta, \alpha).$$

This completes the proof.

References

1. Amer, A. and Darus, M., On some propertis for new generalized derivative operator, Jordan Journal of Mathematics and Statistics (JJMS),4(2), 91-101, (2011).
2. Amer, A.A , Almasri. H, and Darus, M., On neighborhoods of certain classes of p-valent functions with negative coefficients defined by a new extended generalized derivative operator, Acta Universitatis Apulensis, 78(5), 69-79, (2024).
3. Al-Shaqsi, K. and Darus, M., An operator defined by convolution involving polylogarithms functions, J. Math. Stat, 4(1), 46-50,(2008).
4. Al-Abbadi, M. H, and Darus, M., Differential Subordination for new generalized derivative operator, Acta Universitatis Apulensis, 20, 265 280, (2009).
5. AL-Oboudi, F. M, On univalent functions defined by a generalised Salagea Operator, Int. J. Math. Math. Sci. 27, 1429-1436, (2004).
6. Aouf, M. K.; Mostafa, A. O. ; El-Hawsh, G. M. Inclusion properties for classes of p−valent functions associated with linear operator, Afrika Matematika 2024, 1-11, (2024).
7. Aouf, M. K, On a class of p-valent close – to – convex functions od order β and type α , International Journal of Mathematics and Mathematical Sciences, 11(2), 259-266, (1988).
8. Ahmed, A. R. Some applications of Catas operator to p-valent starlike functions, Surman Journal for Science and Technology,401-410, (2024).
9. Bulut, S., On a class of analytic and multivalent functions with negative coefficients defined by Al-Oboudi differential operator, STUDIA UNIV. "BABES¸–BOLYAI", MATHEMATICA ,(2010).
10. Bulboaca, T., Differential Subordinations and Superordinations, Pecent Results, House of Scientific Book Publ., Cluj-Napoca, (2005).
11. Catas, A., On a Certain Differential Sandwich Theorem Associated with a New Generalized Derivative Operator, General Mathematics. 4 ,83-95, (2009).
12. Catas, A., On certain classes of p-valent functions defined by multiplier transformations, Proceedings of the International symposium on Geometric function theory and Applications, 241- 250, (2008).
13. Eenigenburg. P, Miller. S, Mocanu. P, and Reade. M, On a Briot-Bouquet differential subordination, General Enequalities 3, I. S. N. M., Vol. 64, Birkh¨auser Verlag, Basel , 339-348, (1983).
14. Eker , S., Seker. B, On a class of multivalent functions defined by Salagean operator, General Mathematics. 15, 154-163, (2007).

15. Goodman, A. W., On the Shwarz- Christoffel transformation and p-valent functions, Trans. Amer. Math. Soc.,204-223, (1950).
16. Libera, R. J., Some radius of convexity problems, Duke Mathematical Journal, 31(1), 143-158, (1964).
17. Miller. S, Mocanu. P, Second-order differential inequalities in the complex plane, Journal of Mathematical Analysis and Applications, 65(2), 289-305, (1978).
18. Miller. S, Mocanu. P, Differential Subordination ,Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York and Basel,(2000).
19. Miller. S, Mocanu. P, Differential subordinations and univalent functions, Michigan Math. J., 28 no. 2, 157-171, (1981).
20. Mishra, A. K, Sahu, P., Closed convex hull of the family off multivalently close –to-convex functions of order β, Rendi Conti del Circolo Mathematico di Palermo., 209-222, (1999).
21. Munasser, B. M, Mostafa, A. O. , Sultan, T, Nasser, A., EL-Sherbeny and Madian, S. M., Inclusion properties for classes of p-valent functions, Journal of Function Spaces, 1-6, (2024).
22. Owa, S., On certain classes of p-valent functions with negative coefficients, Bull. Belg. Math. Soc. Simon stevin., 385-402, (1985).
23. Partil, D. A. , Thakare, N,K, On convex hulls and extreme points of p- valent starlike and convex classes with applications, Bull. Math. Soc. Sci. Math. R. S., Roumanie(N.S.)., 145-160, (1983).
24. Ruscheweyh, S., New criteria for univalent functions, Proc. Amer.Math. Soc. Vol. 49,109-115, (1975).
25. Raina, R. K., and Srivastava, H. M., Inclusion and neighborhood properties of some analytic and multivalent functions, Journal of Inequalities in Pure and Applied Mathematics, 7(1), (2006).
26. Salagean, G. S, Subclasses of univalent functions, Lecture Notes in Math. (Springer-Verlag), 1013, 362-372, (1983).
27. Silverman, H., Neighborhoods of classes of analytic functions. Far East J. Math. Sci. 3 , 165-169, (1995).
28. Srivastava, H.M., Aouf, M.K., EL-Ashwah, R.M.: Some inclusion relationships associated with a certain class of integral operators. Asian-European J. Math. 3(4), 667–684 (2010).

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.